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MATHEMATICS

On Certain Relations Between Euler-Knopp's and Cesàro's Summability Methods with Respect to the Orthonormal Series

by

J. MEDER

Presented by ST. MAZUR on June 26, 1959

1. In the present paper we shall be concerned with the Euler-Knopp summability method of orthonormal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x), \tag{1}$$

which are expansions of functions with a square integrable in $\langle 0, 1\rangle$, i. e. such, that

$$\sum_{n=0}^{\infty} a_n^2 < \infty. \tag{2}$$

We are especially interested in investigating the relation between Euler-Knopp's and Cesàro's summability methods with respect to the orthonormal series. It is well-known, in the theory of numerical series that the methods (E, q) of Euler-Knopp and (C, r) of Cesàro are not equivalent. However, as we have proved in a recently published paper ([1], Theorem 3, p. 144), if the orthonormal series (1) satisfies condition (2) and is summable by the (E, q)-method for a certain $q > 0$, then it is summable almost everywhere by the (C, r)-method for every $r > 0$. Furthermore, we then assumed, that there exists an orthonormal series, summable by the (C, r)-method in interval $\langle 0, 1\rangle$ and not summable by the (E, q)-method for any set of positive measure. Hereafter is the solution of the above problem.

The following theorem, due to Tandori ([2], Theorem I, pp. 63-83), is fundamental for further considerations:

Let $\{a_n\}$ be a positive, non-increasing, numerical sequence, satisfying the condition

$$\sum_{n=2}^{\infty} a_n^2 \log^2 n = \infty . \tag{3}$$

There exists then an (even uniformly bounded) orthonormal system $\{\Phi_n(x)\}$ (dependent on the sequence $\{a_n\}$), for which the orthonormal series

$$\sum_{n=2}^{\infty} a_n \Phi_n(x) \tag{4}$$

is divergent everywhere in the fundamental interval. This test, the proof of which is very complicated and based on a different idea, is a generalization of the Radamacher-Menchoff theorem.

Taking this result for known, we now formulate

THEOREM 1. *With respect to the summability of orthonormal series* (1), *satisfying condition* (2), *the method* (C, r) *of Cesàro is essentially (i. e. omitting a set of measure zero) weaker than the* (E, q)*-method of Euler-Knopp, for all values of* r *and* q.

In order to prove this Theorem we shall now consider the sequence $b_n\}$, defined in the following way:

$$b_n = 1/\sqrt{n \log n}\,(\log\log n)^2 \quad \text{for} \quad n \geqslant 3^3$$

and

$$b_n = 1 \quad \text{for} \quad n = 2, 3, \ldots, 3^3 - 1 .$$

As may easily be verified, this sequence satisfies all the conditions of Tandori's test and $\sum^{\infty} b_n^2 < \infty$. Now, in view of the above mentioned test, we may construct to the so defined sequence $\{b_n\}$ such a uniformly bounded orthonormal system $\{\Psi_n(x)\}$, that the series

$$\sum_{n=2}^{\infty} b_n \Psi_n(x) \tag{5}$$

will be divergent everywhere in $\langle 0, 1\rangle$.

On the other hand, since $\sum b_n^2 (\log\log n)^2$ is convergent, it follows by virtue of the Kaczmarz-Menchoff theorem, that series (5) is almost everywhere summable by the (C, r)-method for every $r > 0$.

We shall now wish to show that this series is not summable by the (E, q)-method for any $q > 0$ and the value of $x \in \langle 0, 1\rangle$. If even series (5) should be summable by the (E, q)-method for a certain value of $q > 0$ and $x \in \langle 0, 1\rangle$, then taking into account that

$$\sqrt{n}\, b_n \Psi(x) = 0(1)/\sqrt{\log n}\,(\log\log n)^2 = o(1) ,$$

as $n \to \infty$, and applying the well-known principal Tauberian theorem ([3], Theorem 157, p. 221) we could conclude, that series (5) should be convergent, contrary to the Tandori fundamental test.

2. Proceeding to formulate the next theorem, we shall first introduce the following notations:

We denote by $T(s_n)$ the n-th transform of sequence $\{s_n\}$ by the method T. $T_1 T_2$ will indicate the method consisting in applying first the method T_2 to the given sequence $\{s_n\}$ and subsequently the method T_1 to the sequence of transforms thus obtained. In a similar way we define the method $T_1 T_2 T_3$. We shall write C^r for the (C, r)-method of Cesàro and E^q for the (E, q)-method of Euler-Knopp. The generalized limit of sequence $\{s_n\}$ by the method $E^q C^r$ we denote by $E^q C^r - \lim s_n$.

We observe that the transforms $E^q C^r(s_n)$ and $C^\alpha C^\beta(s_n)$ are commutable ([4], § 63). Moreover, from the classical results of Hausdorff [5]), it follows, that $C^\alpha C^\beta \sim C^{\alpha+\beta}(\alpha, \beta, \alpha+\beta > -1)$.

It is known from the theory of numerical series, that if $C^r\text{-}\lim s_n$ exists, then $E^q C^r\text{-}\lim s_n$ also exists, and $C^r\text{-}\lim s_n = E^q C^r\text{-}\lim s_n$. But the existence of $E^q C^r\text{-}\lim s_n$ does not imply that of $C^r\text{-}\lim s_n$, for any r and q positive.

In an earlier paper ([1], Theorem 5, p. 146) we have proved the following theorem: if the orthonormal series (1) satisfies condition (2) and $CE\text{-}\lim s_n(x)$ exists almost everywhere, then $C\text{-}\lim s_n(x)$ exists almost everywhere, and $CE\text{-}\lim s_n(x) = C\text{-}\lim s_n(x)$.

The last theorem may be generalized as follows:

THEOREM 2. *If the orthonormal series* (1) *satisfies condition* (2), $s_n(x)$ *denote its n-th partial sums, and* $E^q C^r\text{-}\lim s_n(x)$ *exists almost everywhere for a certain r and q positive, then* $C^r\text{-}\lim s_n(x)$ *exists almost everywhere and* $E^q C^r\text{-}\lim s_n(x) = C^r\text{-}\lim s_n(x)$.

In order to prove this theorem, we first observe that

$$C(E^q C^r) = C(C^r E^q) \sim C^{1+r} E^q = E^q C^{1+r}. \tag{6}$$

We may thus suppose that $r > 1$.

Let us consider the series

$$\sum_{n=1}^{\infty} [\sigma_n^r(x) - \sigma_{n-1}^r(x)], \tag{7}$$

where $\sigma_n^r(x)$ denote the n-th Cesàro means of order r of series (1).

Without loss of generality, we may assume in series (1) $a_0 = 0$. Then the n-th partial sums of series (4) are $\sigma_n^r(x)$. Suppose that the conditions of Theorem 2 are satisfied. Then, since

$$E^q C^r\text{-}\lim s_n(x) = E^q\text{-}\lim \sigma_n^r(x), \tag{8}$$

it follows that series (7) is summable almost everywhere by the (E, q)-method.

By virtue of the Kaczmarz-Zygmund lemma (see [6], pp. 184 and 185) and condition (2), the series

$$\sum_{n=1}^{\infty} n[\sigma_n^r(x) - \sigma_{n-1}^r(x)]^2$$

is almost everywhere convergent.

Hence, we have

$$\sigma_n^r(x) - \sigma_{n-1}^r(x) = o(n^{-1/2}),$$

for almost every $x \in \langle 0, 1 \rangle$.

According to this relation and the principal Tauberian theorem, it follows that series (7) is almost everywhere convergent. Thus, series (1) is almost everywhere summable by the (C, r)-method and by (8) $C^r\text{-}\lim s_n(x) = E^q C^r\text{-}\lim s_n(x)$, which completes the proof.

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MATHÉMATIQUE

Le théorème d'unicité de la solution du système parabolique des équations linéaires avec les coefficients höldériens

par

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Présenté par K. KURATOWSKI le 30 juin 1959

1. On traite ici la question d'unicité pour le système parabolique des équations linéaires avec les coefficients höldériens [1] dans la classe des solutions du système, soumis aux conditions (Z_1) et (Z_4), et dans le produit cartésien de l'espace euclidien E — à n dimensions — et du temps: $0 < t < T$, $T = \text{const.}$: $(E \times (0, T))$. Les notations sont conformes à celles des travaux [1] et [2].

Soient $N > 1$ et $M > 2$ des nombres entiers; $X = X(x_1, \ldots, x_n)$.

$$(1) \qquad \hat{\psi}^{(\alpha)}(u_1, \ldots, u_N) = \sum_{1 \leqslant j \leqslant N} \sum_{0 \leqslant k_1 + \ldots + k_n \leqslant M} A_{\alpha,j}^{k_1, \ldots, k_n}(X, t) \frac{\partial u_j^{k_1, \ldots, k_n}}{\partial x_1^{k_1}, \ldots, \partial x_n^{k_n}} - \frac{\partial u_\alpha}{\partial t} = 0, \qquad \alpha = 1, \ldots, N,$$

$u_1(X, t), \ldots, u_N(X, t)$; $X \in E$, $0 < t < T$, sont les composantes de la solution du système (1). On fait des hypothèses suivantes:

(Z_1). Les dérivées spaciales d'ordre M et la première dérivée par rapport à t sont continues dans $(E \times (0, T))$; en plus on pose:

$$\sum_{1 \leqslant j \leqslant N} \sum_{0 \leqslant k_1 + \ldots + k_n \leqslant M} \int_0^T \int_E \left| \frac{\partial u_j^{k_1 + \ldots + k_n}(X, t)}{\partial x_1^{k_1}, \ldots, \partial x_n^{k_n}} \right| dX\, dt \leqslant C_1 = \text{const}\ ^*).$$

(Z_2). Les coefficients du système (1) sont continus dans $(E \times \langle 0, T \rangle)$ et sont absolument bornés par la constante positive C_2; tous les coefficients sont höldériens par rapport à la variable spaciale (avec l'exposant

*) Au cas de l'intégrale de Riemann il faudrait supposer outre cela que toutes ces dérivées convergent vers zéro pour $|X| \to \infty$.

de puissance h), et les coefficients des dérivées d'ordre M sont de plus höldériens par rapport à la variable t (avec l'exposant de puissance h').

(Z_3). Le système (1) est parabolique selon Petrovsky [3] (v. aussi [1]), à savoir les radicaux $\lambda_\alpha(X, t)$ de l'équation de Petrovsky:

$$\det_{\alpha,j}\Big| \sum_{k_1+\ldots+k_n=M} A_{\alpha,j}^{k_1,\ldots,k_n}(X,t)(is_1)^{k_1}\ldots(is_n)^{k_n} - \delta_\alpha^j \lambda \Big| = 0$$

où:

$$\delta_\alpha^j = \begin{matrix} 1, & \alpha = j \\ 0, & \alpha \neq j, \end{matrix}$$

$s_1^2+\ldots+s_n^2=1$, $s_1, \ldots, s_n$ étant réels sont soumis à l'inégalité:

$$\operatorname{Re}\lambda_\alpha(X,t) < -\delta, \ \delta > 0, \qquad \alpha = 1, \ldots, N.$$

(Z_4). Les fonctions initiales des composantes de la solution $(u_1, \ldots, u_N)$ sont identiquement nulles, c'est à dire:

$$u_1(X,0) \equiv 0, \ldots, u_N(X,0) \equiv 0, \ X \in E.$$

On démontrera qu'une solution du système (1), dont les coefficients sont conformes aux hypothèses (Z_2) et (Z_3), est identiquement nulle, dès qu'elle accomplit les hypothèses (Z_1) et (Z_4), c'est à dire:

$$u_1(X,t) \equiv 0, \ldots, u_N(X,t) \equiv 0, \qquad X \in E, \qquad 0 \leqslant t \leqslant T.$$

Il en résulte que deux solutions du système (1) dont la différence converge vers zéro, pour $t \to 0$, sont partout identiques, dès qu'elles satisfont à l'hypothèse (Z_1).

La démonstration, comme chez Petrovsky, sera basée sur le système adjoint au système (1).

Mais chez Petrovsky les coefficients du système ne dépendent que du temps t, tandis que les coefficients du système (1) dépendent aussi de la variable spaciale; ils devraient donc se trouver dans le système adjoint sous le signe de la différentiation, [4], [5], mais on a pas supposé l'existence de leurs dérivées. On doit donc les remplacer par les coefficients différentiables partout. Ces nouveaux coefficients devrons constituer une assez bonne approximation de ceux du système (1) dans un domaine borné de l'espace E, contenant l'origine (dite partie centrale).

2. La construction des coefficients différentiables. On cherche les fonctions $\widetilde{A}^{k_1,\ldots,k_n}(X,t)$, $0 \leqslant k_1+\ldots+k_n \leqslant M$ $\alpha, j = 1, \ldots, N$, continues pour chaque entier $m = 1, 2, \ldots$ dans $(E \times \langle 0, T\rangle)$, de modules bornés, höldériens comme les coefficients correspondants $A_{\alpha,j}^{k_1,\ldots,k_n}(X,t)$ du système (1) dans E, ou pour le cas $k_1+\ldots+k_n = M$ dans $(E \times \langle 0, T\rangle)$. Ces coefficients nouveaux sont, en outre, différentiables par rapport à la

variable spaciale au moins $(M+1)$ fois, les derivées étant continues et bornées; $m = \text{const}$. Aussi, ces coefficients doivent assurer la parabolicité du système suivant:

$$(1.2) \qquad \widetilde{\psi}_m^{(a)}(\widetilde{u}_1, \ldots, \widetilde{u}_N) = \sum_{1 \leqslant j \leqslant N}^{0 \leqslant k_1 + \ldots + k_n \leqslant M} \widetilde{A}_{a,j}^{k_1 \ldots k_n, m}(X, t) \frac{\partial \widetilde{u}_j^{k_1 + \ldots + k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} - \frac{\partial \widetilde{u}_a}{\partial t} = 0,$$

$$a = 1, \ldots, N, \quad m = \text{const}, \quad (X, t) \in (E \times (0, T\rangle), \ [5].$$

Dans ce but on construit des coefficients auxiliaires, continus et periodiques $\bar{A}_{a,j}^{k_1, \ldots, k_n, R}(X, t)$ qui sont égaux aux coefficients du système (1) dans la partie centrale de l'espace E. Ils sont hölderiens comme les coefficients du système (1). Chaque coefficient cherché $\widetilde{A}_{a,j}^{k_1, \ldots, k_n, m}(X, t)$ est la m-ièmne moyenne de Féjer du coefficient auxiliaire $\bar{A}_{a,j}^{k_1, \ldots, k_n, R}(X, t)$. On démontre qu'un tel coefficient $\widetilde{A}_{a,j}^{k_1, \ldots, k_n, m}(X, t)$ est circa höldérien comme le coefficient correspondant du système (1), c'est à dire: les exposants de puissance ne changent pas. La constante C est multipliée par 3, indépendemment de l'entier m. On a, en outre:

$$(1',2) \qquad |\widetilde{A}_{a,j}^{k_1, \ldots, k_n, m}(X, t)| \leqslant 3C_2.$$

On démontre aussi, que $\operatorname{Re}\widetilde{\lambda}_a^m(X, t)$ désignant la partie réelle du radical de l'équation de Petrovsky pour le système (1.2), on a:

$$(2.2) \qquad \operatorname{Re}\widetilde{\lambda}_a^m(X, t) < -\delta/2, \quad \delta > 0, \quad a = 1 \ldots N,$$

dès que:

$$(3.2) \qquad m \geqslant L(R, \delta),$$

où $L(R, \delta)$ est un entier qui ne dépend que de R, et du système (1).

3. Le système adjoint au système (1.2) sera le suivant:

$$(1.3) \qquad {}^*\widetilde{\psi}_m^{(a)}(\widetilde{v}_1, \ldots, \widetilde{v}_N)$$

$$= \sum_{1 \leqslant j \leqslant N}^{0 \leqslant k_1 + \ldots + k_n \leqslant M} (-1)^{k_1 + \ldots + k_n} \frac{\partial^{k_1 + \ldots + k_n} \widetilde{v}_j \widetilde{A}_{ja}^{k_1, \ldots, k_n, m}(X, t)}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} + \frac{\partial \widetilde{v}_a}{\partial t} = 0$$

$a = 1, \ldots, N$. $\widetilde{v}_1, \ldots, \widetilde{v}_N$ sont les composantes de la solution du système (1.3).

Le système (1.3) est parabolique car l'indice m est soumis à l'inégalité (3.2), [5]. Aussi, il résulte du travail [5], qu'il possède des solutions absolument intégrables avec leurs derivées jusqu'à l'ordre du système.

LEMME (1.3). *Si les solutions des systèmes* (1.2) *et* (1.3) *avec leurs dérivées jusqu'à l'ordre du système sont bornées et absolument intégrables dans* $(E\times(t_1\leqslant t\leqslant t_2))$, $(t_1, t_2)\in(0, T\rangle$ *alors*:

$$\sum_{\alpha=1}^{N}\int_E \widetilde{u}_\alpha(X,t)\widetilde{v}_\alpha(X,t)\,dX=\text{const},\qquad t_1\leqslant t\leqslant t_2. \tag{2.3}$$

La démonstration du Lemme (1.3) résulte de la formule:

$$\sum_{\alpha=1}^{N}\int_{t_1}^{t_2}\int_E \widetilde{v}_\alpha\widetilde{\varphi}_m^{(\alpha)}(u_1,\ldots,u_N)\,dX\,dt=0,\qquad 0<t_1<t_2<T,$$

après les intégrations par parties convenables; la démonstration est un peu modelée sur celle du travail [4].

Soient maintenant (3.3) les N systèmes des solutions fondamentales du système (1.2):

$$\{\widetilde{\Gamma}^m_{\alpha,j}\}_{\alpha=1,\ldots,N};\qquad j=1,\ldots,N \tag{3.3}$$

et

$$\{{}^*\widetilde{\Gamma}^m_{\alpha,j}\}_{\alpha=1,\ldots,N};\qquad j=1,\ldots,N \tag{4.3}$$

du système (1.3).

On sait [6] que si les fonctions $U_\alpha(X,t)$ et $V_\alpha(X,t)$, $\alpha=1,\ldots,N$ sont continues et absolument intégrables dans $(E\times\langle 0, T\rangle)$, alors les intégrales:

$$\mathfrak{J}^m_\alpha(X,t,\tau)=\int_E\sum_{j=1}^{N}U_j(Y,\tau)\widetilde{\Gamma}^m_{\alpha,j}(X,t;Y,\tau)\,dY,\quad t>\tau,\ \alpha=1,\ldots,N \tag{5,3}$$

et

$${}^*\mathfrak{J}^m_\alpha(X,t,\tau)=\int_E\sum_{j=1}^{N}V_j(Y,\tau)\,{}^*\widetilde{\Gamma}^m_{\alpha,j}(Y,\tau;X,t)\,dY,\quad t<\tau,\ \alpha=1,\ldots,N \tag{5',3}$$

sont des solutions des systèmes (1.2) et (1.3) respectivement, en tant que fonctions du point (X,t). Outre cela, elles sont soumises à la condition (Z_1) et on sait [2], que:

$$\lim_{t\to\tau+}\mathfrak{J}^m_\alpha(X,t,\tau)=U_\alpha(X,\tau),\qquad \alpha=1,\ldots,N;\quad 0\leqslant\tau<T,\ t>\tau, \tag{6.3}$$

$$\lim_{t\to\tau-}{}^*\mathfrak{J}^m_\alpha(X,t,\tau)=V_\alpha(X,\tau),\qquad \alpha=1,\ldots,N,\ 0<\tau\leqslant T,\ t<\tau. \tag{6'.3}$$

C'est pourquoi $U_\alpha(X,\tau)$ est nommée la valeur initiale de l'intégrale $\mathfrak{J}^m_\alpha(X,t,\tau)$, et $V_\alpha(X,\tau)$ – la valeur finale de l'intégrale ${}^*\mathfrak{J}^m_\alpha(X,t,\tau)$. L'application du Lemme (1.3) aux systèmes de fonctions $\{\mathfrak{J}^m_\alpha\}_{\alpha=1,\ldots,N}$ et $\{{}^*J^m_\alpha\}_{\alpha=1,\ldots,N}$ pour $0<t_1<t_2\leqslant T$ et les suppositions comme dans (5.3) et (5'.3) nous conduisent à la formule:

$$\sum_{\alpha=1}^{N}\int_E \mathfrak{J}^m_\alpha(X,t_2,t_1)V_\alpha(X,t_2)\,dX=\sum_{\alpha=1}^{N}\int_E {}^*\mathfrak{J}^m_\alpha(X,t_1,t_2)\,U_\alpha(X,t_1)\,dX. \tag{7,3}$$

4. L'application de la formule (7.3) à la démonstration de l'unicité. LEMME (1,4). *A chaque solution* $(u_1, \dots, u_N)$ *du système* (1) *soumis aux hypothèses* $(Z_1) \dots (Z_4)$ *et chaque nombre* $\varepsilon > 0$ *correspond un tel indice positif* $L(\varepsilon)$ *et une telle solution du système* (1.3): $(\tilde{v}_1, \dots, \tilde{v}_N)$, *qu'il résulte de l'inégalité*: $m \geqslant L(\varepsilon)$, *que*

$$(1,4) \qquad |r_m| = \left| \int_0^\tau \sum_{\alpha=1}^N \int_E \tilde{v}_\alpha \sum_{1 \leqslant j \leqslant N}^{0 \leqslant k_1 + \dots + k_n \leqslant M} [A_{\alpha j}^{k_1, \dots, k_n}(X,t) - \tilde{A}_{\alpha j}^{k_1, \dots, k_n, m}(X,t)] \times \right.$$
$$\left. \times \frac{\partial u_j^{k_1 + \dots + k_n}}{\partial x^{k_1} \dots \partial x_n^{k_n}} dX\, dt \right| \leqslant \frac{\varepsilon}{2}.$$

La solution $(\tilde{v}_1, \dots, \tilde{v}_N)$ dont il s'agit est:

$$(2,4) \qquad \tilde{v}_\alpha = {}^*\tilde{J}_\alpha^m(X,t,\tau), \quad t < \tau,$$

avec la fonction finale, absolument intégrable dans E, définie comme il suit:

$$(3,4) \qquad V_\alpha(X,\tau) = \begin{cases} 0 \ , & \alpha = 2, \dots, N \\ v(x), & \alpha = 1, \end{cases}$$

où $v(X)$ est continue pour $X \in E$, $v(X) = 0$, $X \bar{\in} K_0$; $0 \leqslant v(X) \leqslant 2$, $X \in K_0$, $X \in K_0$, et

$$(3',4) \qquad \int_{K_0} v(X)\, dX = 1,$$

où $K_0 \in E$ est un domaine qui sera précisé au § 5. Il résulte des définitions (3,4), (2,4), (5',3) et de l'égalité (6',3), que

$$(4,4) \qquad \tilde{v}_\alpha(X,t,\tau) \underset{t \to \tau}{\to} \begin{cases} 0 \ , & \alpha = 2, \dots, N \\ v(X), & \alpha = 1; \end{cases} \qquad t < \tau.$$

On remplace la fonction initiale $U_\alpha(X,t)$ dans la formule (5,3) par l'expression:

$$(5,4) \qquad \overline{U}_\alpha(X,t) = \sum_{1 \leqslant j \leqslant N}^{0 \leqslant k_1 + \dots + k_n \leqslant M} [A_{\alpha j}^{k_1, \dots, k_n}(X,t) - \tilde{A}_{\alpha,j}^{k_1, \dots, k_n, m}(X,t)] \frac{\partial u_j^{k_1 + \dots + k_n}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}},$$

qui est absolument intégrable dans E. $\alpha = 1, \dots, N$, $0 < t \leqslant \tau$ et on reçoit au lieu de (5,3):

$$(5',4) \qquad \tilde{J}_\alpha^m(X,t,\tau) = \int_E \sum_{j=1}^N \overline{U}_j(Y,\tau) \tilde{\Gamma}_{\alpha,j}^m(X,t;Y,\tau)\, dY, \quad \alpha = 1, \dots, N, \ \tau < t.$$

Il résulte de ces définitions, de l'égalité (7,3) pour $t_1 = t$, $t_2 = \tau$, que

$$(6,4) \qquad r_m = \int_0^\tau \sum_{\alpha=1}^N \int_E \tilde{v}_\alpha(X,t,\tau)\, \overline{U}_\alpha(X,t)\, dX\, dt$$
$$= \int_0^\tau \sum_{\alpha=1}^N \int_E \tilde{J}_\alpha^m(X,\tau,t)\, V_\alpha(X,\tau)\, dX\, dt = \int_0^\tau \int_{K_0} v(X) \tilde{J}_1^m(X,\tau,t)\, dX.$$

Donc, on obtient, vu les formules (4,4), (5′,4) et (5,4) l'égalité:

$$(7,4)\qquad r_m = \sum_{1\leqslant j\leqslant N}^{0\leqslant k_1+\ldots+k_n\leqslant M} \int_0^\tau \int_E \frac{\partial u_j^{k_1+\ldots+k_n}(Y,t)}{\partial y_1^{k_1}\ldots\partial y_n^{k_n}} \times$$

$$\times \int_{K_0} \sum_{\nu=1}^{N} [A_{\nu j}^{k_1,\ldots,k_n}(Y,t) - \tilde{A}_{\nu j}^{k_1,\ldots,k_n,m}(Y,t)] v(X) \tilde{\Gamma}_{1,\nu}^{m}(X,\tau;\, Y,t)\, dX\, dY\, dt\,.$$

Désignons par $I_j^m(Y,t,\tau)$ l'intégrale de l'équation donnée ci-dessus étendue sur l'ensemble K_0. On trouve d'après l'hypothèse (Z_2), du § 2, en particulier de la formule (1′,2) et du travail [2], que:

$$(8,4)\qquad |I_j^m(Y,t,\tau)| \leqslant C_3 = \text{const}$$

$$j = 1,\ldots,N,\ 0 \leqslant k_1+\ldots+k_n \leqslant M\,,\ (Y,t)\,\epsilon\,(E\times\langle 0,\tau\rangle)\,,$$

pour les indices m conformes à l'inégalité (3,2). La constante C_3 ne depend que des constantes $\delta/2$, C_2, $3C$, h, h'(v. (Z_2), (Z_3)), donc C_3 ne dépend pas de m. Il résulte de l'hypothèse (Z_1) que si le rayon R de la sphère $K(O,R_0)$ est assez grand et si $\bar{K}(0,R_0)$ est son complément, alors on a

$$(9,4)\qquad \sum_{1\leqslant j\leqslant N}^{0\leqslant k_1+\ldots+k_n\leqslant M} \int_0^\tau \int_{\bar{K}(0,R_0)} \left|\frac{\partial u_j^{k_1+\ldots+k_n}(X,t)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right| dX\, dt \leqslant \frac{\varepsilon}{4C_3}\,,\qquad \varepsilon > 0\,.$$

Posons dans le § 2, $R = 2R_0$ et le nombre $L\ (\delta, 2R_0)$ assez grand pour que l'inégalité (3,2) implique l'inégalité (2,2). L'inégalité (8,4) sera alors satisfaite. Celle-ci avec (9,4) et (1′,2) permet de remplacer l'expression r_m (voir (1,4)), à la constate $\varepsilon/4$ près, par l'expression:

$$(10,4)\qquad r_m^* = \sum_{1\leqslant j\leqslant N}^{0\leqslant k_1+\ldots+k_n\leqslant M} \int_0^\tau \int_{K(0,R_0)} \frac{\partial u_j^{k_1+\ldots+k_n}(Y,t)}{\partial y_1^{k_1}\ldots\partial y_n^{k_n}}\, I_j^m(Y,t,\tau)\, dY\, dt\,.$$

Mais, d'après le travail [2], ainsi que d'après la définition des coefficients $\tilde{A}_{\nu j}^{k_1,\ldots,k_n,m}(Y,t)$ nous avons:

$$(10',4)\qquad \lim_{m\to\infty} I_j^m(Y,t,\tau) = 0\,,$$

d'où, des propositions (8,4), (9,4), (10,4) et du théorème de Lebesgue il résulte que

$$(10'',4)\qquad \lim_{m\to\infty} r_m^* = 0\,,$$

ce qui prouve qu'on a, vu (9,4), l'indice $L_1(\varepsilon)$, etant assez élévé, pour $m \geqslant L_0 = \text{Max}[L_1(\varepsilon), L(\delta, 2R_0)]$ les inégalités:

$$(11,4)\qquad |r_m| \leqslant \varepsilon/2\,,\ |r_m^*| \leqslant \varepsilon/4\,.$$

5. La démonstration du Théorème d'unicité. On suppose, par contradiction, sous les hypothèses $(Z_1), \ldots, (Z_4)$ que la solution $(u_1, \ldots, u_n)$ du système (1) n'est pas identiquement nulle dans $(E \times \langle 0, T \rangle)$, par exemple que l'on ait $u_1(X_0, t) \not\equiv 0$; soit $u_1(X_0, \tau) > 2\varepsilon$, $\varepsilon > 0$, $\tau > 0$. Soit $K_0(X_0) \in E$ l'entourage de X_0, où

$$u_1(X, \tau) > \epsilon, \quad X \in K_0(X_0). \tag{1,5}$$

Posons dans toutes les formules du § 4: $K_0 = K_0(X_0)$. On trouve à cause de

$$\sum_{\alpha=1}^{N} \int_0^{\tau} \int_E \tilde{v}_\alpha(X, t, \tau) \hat{\psi}^{(\alpha)}_{X,t}(u_1, \ldots, u_N)\, dX\, dt = 0, \tag{2,5}$$

et selon le Lemme (1,4) on a, pour $m > L_0$:

$$\sum_{\alpha=1}^{N} \int_0^{\tau} \int_E \tilde{v}_\alpha(X, t, \tau) \sum_{1 \leqslant j \leqslant N}^{0 \leqslant k_1 + \ldots + k_n \leqslant N} \tilde{A}^{k_1, \ldots, k_n m}_{\alpha j}(X, t) \frac{\partial u_j^{k_1 + \ldots + k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} dX\, dt - \tag{3,5}$$

$$- \sum_{\alpha=1}^{N} \int_0^{\tau} \int_E \tilde{v}_\alpha(X, t, \tau) \frac{\partial u_\alpha}{\partial t}\, dX\, dt + \theta\varepsilon/2 = 0. \qquad |\theta| \leqslant 1$$

d'où par les intégrations par parties convenables et à cause de (Z_4) il résulte:

$$\sum_{\alpha=1}^{N} \int_E V_\alpha(X, \tau) u_\alpha(X, \tau)\, dX + \theta\varepsilon/2 = 0, \tag{4,5}$$

ce qui amène, à cause des formules (2,4), ..., (4,4) et (1,5), à la contradiction, puisqu'on aurait:

$$\varepsilon + \theta\varepsilon/2 = 0, \qquad \varepsilon > 0, \qquad |\theta| \leqslant 1.$$

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MATHEMATICS

A Generalization of the Gauss-Bonnet Formula

by

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In [3] and [4] M. N. Stavroulakis generalized the known Gauss-Bonnet formula of differential geometry to the case of surfaces with vertical or "rebroussement" points. The aim of this paper is to generalize the Gauss-Bonnet formula to a wider class of surfaces (see Theorem 3).

1. Let L be a ray issuing from the limit point $p \in X \subset E_3$. L is called *the demitangent to the set X in p*, if for every circular cone C with the vertex p and the axe L there exists a point $q \in X$ $(q \neq p)$ contained inside of C [2].

2. The set $P \subset E_3$ is called *a surface*, if P is locally homeomorfic with E_2.

3. The set $\Delta \subset E_3$ is called *a surface sheet*, if Δ is homeomorfic with a plain domain.

4. If S is the union of all demitangents to the closure $\overline{P}$ of the surface P in the point $p \in \overline{P}$ and $\dim S \leqslant 2$, then it will be called *the cone tangent to $\overline{P}$ in p*.

5. The point $p \in \overline{P}$ will be called *a point of I kind* if a cone tangent to $\overline{P}$ in p reduces to a ray; p will be called *a point of II kind* if there exists a neighbourhood $U(p) \subset \overline{P}$ such that a) a cone S tangent to $\overline{P}$ in p is homeomorfic with $U(p)$ and b) $U(p)-p$ may be represented as a union of disjoint arcs L_t issuing from p such that to every generatrix T of S corresponds one, and only one, arc L_t tangent to T; p will be called *a point of III kind* if there exists a neighbourhood $U(p) \subset \overline{P}$ which may be represented as a closure of a union of disjoint sheets Δ_i, $i = 1, \ldots, k$ such that $\bigcap_{i=1}^{k} \overline{\Delta}_i = p$ and p is the point of I or II kind on each of the sets $\overline{\Delta}_i$.

The cone S tangent to $\overline{P}$ in the point p of I, II or III kind will be called *the cone of I, II or III kind*, respectively.

6. Let S be a cone with the vertex p. The set $S \cap R$, where R is a sphere of radius 1 and the centre p, will be called *a characteristic curve of the cone* S *).

7. Let Δ be a surface sheet. Let $p \in \Delta$ and let $L', L'' \subset \Delta$ be two arcs with the common origin p. We shall define *the angle* α between L' and L''.

Let V be a neighbourhood of p in Δ such that $V-(L' \cup L'')$ has two components Δ^1 and Δ^2.

In each of Δ^i $(i = 1, 2)$ we choose a sequence of arcs $L_j^{(i)}$ $(j = 1, \ldots, n)$ such that:

(1) the point p is the only common point of all $L_j^{(i)}$;

(2) $L_1^{(i)} = L'$, $L_n^{(i)} = L''$;

(3) the arcs $L_1^{(i)}, \ldots, L_n^{(i)}$ are simply ordered from $L_1^{(i)}$ to $L_n^{(i)}$.

(The existence of such sequences follows from the existence of analogous sequences of rays on the plane).

Let $\Delta_j^{(i)}$ be the subset of $\Delta^{(i)}$ contained between $L_j^{(i)}$ and $L_{j+1}^{(i)}$ $(j=1, \ldots, n-1; i=1,2)$. Let $S_j^{(i)}$ be a cone tangent in p to $\Delta_j^{(i)}$ $(j=1, \ldots, n-1;$ $i = 1, 2)$ and let $\alpha_j^{(i)}$ be the length of the characteristic curve of $S_j^{(i)}$.

$$\alpha \overset{\text{df}}{=} \min_i \sup \sum_{j=1}^{n-1} \alpha_j^{(i)},$$

where sup is taken over all possible sequences $\{L_j^{(i)}\}$ satisfying (1)-(3).

The number

$$\varphi = \sum_{i=1}^{2} \sup \sum_{j=1}^{n-1} \alpha_j^{(i)}$$

will be called *the total angle around the point* p on the sheet Δ (or the total angle around the vertex of a cone tangent in p to Δ).

Let $p \in \mathrm{Fr}\,\Delta$ **) and let $L', L'' \subset \mathrm{Fr}\,\Delta$ be two arcs with the common origin p; let $L_j \subset \Delta$ $(j = 1, \ldots, n)$ be a sequence of arcs satisfying (1)-(3), and let α_j be the length of the characteristic curve of the cone S_j tangent in p to the subset Δ_j of Δ contained between L_j and L_{j+1} $(j = 1, \ldots, n-1)$.

The angle α between L' and L'' is defined by

$$\alpha \overset{\text{df}}{=} \sup \sum_{j=1}^{n} \alpha_j,$$

where sup is taken over all possible sequences $\{L_j\}$ satisfying (1)-(3).

8. The point p will be called *a vertical point* of the closure $\overline{P}$ of the surface P, if $p \in \overline{P}$ and if the cone tangent in p to $\overline{P}$ is not the union of two half-planes.

*) The characteristic curve needs not be connected.

**) Fr denotes the boundary of sheet Δ, i. e. $\mathrm{Fr}\,\Delta = \bar{\Delta} - \Delta$.

9. The regular open arc *) $L \subset P$ will be called *an edge of* P if the cone tangent to P in an arbitrary point of L is a union of two half-planes but not a plane.

10. Regular simple connected sheets (Δ_i) of the class C^2, regular open arcs (L_j) not containing any vertical point of $\overline{P}$, and single points (p_k) will be called *the elementary sets of* $\overline{P}$ (we call them elementary sheets, elementary arcs and elementary points, respectively).

11. The regular sheet $Q(L)$ will be called a *continuation of the sheet* Δ over the arc $L \subset \operatorname{Fr}\Delta$, if $\Delta \cup L \subset Q(L)$ and $\Delta \cup L \neq Q(L)$ **).

In this paper we shall assume that

1°. For every point $p \in \overline{P}$ the union of all the demitangents to $\overline{P}$ in p is a cone the characteristic curve of which is the union of a finite number of arcs of the class C^2.

2°. $\overline{P}$ contains at most a finite number of vertical points.

3°. P contains at most a finite number of disjoint edges such that the integral of the curvature exists for each of them.

4°. Points of P in which the second derivatives of the parametric representation of P are not continuous, form a union of a finite number of points and of a finite number of regular open arcs.

5°. P is orientable.

6°. For every two points $p, q \in P$ there exists a shortest arc $L \subset P$ joining p with q.

From the assumption 2°, 3° and 4° it follows that we can divide the surface P into a finite number of disjoint elementary sets in such a manner that

(4) $\prod_{i_1,i_2} \overline{\Delta}_{i_1} \cap \overline{\Delta}_{i_2}$ is the arc of the division;

(5) $\prod_{j_1,j_2} \overline{L}_{j_1} \cap \overline{L}_{j_2}$ is the point of the division;

(6) every edge, every vertical point and every point or arc mentioned in the assumption 4° is an element of this division.

11. We shall define *a total curvature* $\Omega(X)$ of an elementary set X.

I. If p is a point of the surface P, then

$$\Omega(p) = 2\pi - \varphi\,,$$

where φ is the total angle around p on P.

II. If L is an elementary arc of P, Δ_1, Δ_2 are elementary sheets such that $L = \overline{\Delta}_1 \cap \overline{\Delta}_2$, $\varkappa_g^i(s)$ $(i = 1, 2)$ is the geodetical curvature of L on the sheet $Q_i(L)$ which is a continuation of the sheet Δ_i over L, then

$$\Omega(L) \overset{\text{df}}{=} \int_L [\varkappa_g^1(s) + \varkappa_g^2(s)]\,ds\,.$$

*) A regular arc is such one, which has a continuous tangent straight line.

**) It is easy to prove that for an arbitrary regular arc L on the surface P there exists such a sheet $Q(L)$.

Evidently, $\Omega(L)$ does not depend on the choice of the sheets $Q_i(L)$ – it follows from the definition of geodetical curvature.

III. If Δ is an elementary sheet on the surface P, Γ_i are disjoint regular arcs such that $\Gamma_i \subset \operatorname{Fr}\Delta$, $\overline{\bigcup_i \Gamma_i} = \operatorname{Fr}\Delta$; α_j is the angle between Γ_j and Γ_{j+1} on Δ; $\varkappa_g^i(s)$ is the geodetical curvature of the arc Γ_i on the sheet Q_i which is a continuation of Δ over Γ_i, then

$$\Omega(\Delta) \overset{\text{df}}{=} 2\pi - \sum_i (\pi - \alpha_i) - \sum_j \int_{\Gamma_j} \varkappa_g^i(s)\, ds\,.$$

Let l_ν be the number of ν-dimensional elementary sets of a certain division of the surface P ($\nu = 0, 1, 2$); then

$$\Omega(P) \overset{\text{df}}{=} \sum_{i=1}^{l_0} \Omega(p_i) + \sum_{j=1}^{l_1} \Omega(L_j) + \sum_{k=1}^{l_2} \Omega(\Delta_k)\,.$$

If P is a regular surface then $\Omega(P) = \sum_{k=1}^{l_2} \Omega(\Delta_k) = C(P)$, where $C(P)$ is the curvatura integra, i. e. $C(P) = \iint_P K dS$ (it follows from the Gauss-Bonnet formula).

It is easy to prove that P is "a manifold of bounded curvature" in the sense introduced by A. D. Aleksandrov ([1], p. 492) and that $\Omega(P) = \omega(P)$, where $\omega(P)$ is a curvature in the sense introduced by Aleksandrov ([1], p. 496).

THEOREM 1. *Let Δ be a sheet satisfying* 1^0-5^0, *$\chi(\Delta)$ – the Euler-Poincaré characteristic of Δ, $\Gamma = \operatorname{Fr}\Delta$ – a curve containing n angled points *, Γ_i – disjoint regular arcs such that $\overline{\sum_i \Gamma_i} = \Gamma$, $\varkappa_g^i$ – the geodetical curvature of Γ on the sheet Q_i, which is a continuation of Δ over Γ_i, α_j – the angle between Γ_i, Γ_{j+1} on Δ.*

Then

$$\Omega(\Delta) = 2\pi \cdot \chi(\Delta) - \sum_{j=1}^{n} (\pi - \alpha_j) - \int_\Gamma \varkappa_g(s)\, ds \;\text{**}).$$

Proof. Let Δ_i ($i = 1, \ldots, l_2$), L_j ($j = 1, \ldots, l_2$), p_k ($k = 1, \ldots, l_0$) be elementary sets of the sheet Δ, α_i^ν – interior angles of the curve $\operatorname{Fr}\Delta_i$

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*) An angled point of a curve is a point in which both demitangents form an angle $\neq \pi$.

**) $\int_\Gamma \varkappa_g ds \overset{\text{df}}{=} \sum_i \int_{\Gamma_i} \varkappa_g ds$.

$= \sum_{\nu} \overline{\Gamma_i^{\nu}}$ (i. e. the angles between Γ_i^{ν} and $\Gamma_i^{\nu+1}$ on Δ_i), φ_k – a total angle around p_k.

According to the definition of $\Omega(\Delta_i)$:

$$\sum_{i=1}^{l_2} \Omega(\Delta) = 2\pi l_2 - \sum_{i,\nu} (\pi - \alpha_i^{\nu}) - \sum_{j,\mu} \int_{\Gamma_j^{\mu}} \varkappa_g ds$$

$$= 2\pi l_2 - \pi(2l_1 + n) + \sum_{i=1}^{l_0} \varphi_i + \sum_{i=1}^{n} \alpha_i - \sum_{j,\mu} \int_{\Gamma_j^{\mu}} \varkappa_g ds$$

$$= 2\pi(l_2 - l_1 + l_0) - \sum_{i=1}^{n} (\varkappa - \alpha_i) + \sum_{i=1}^{l_0} \varphi_i - \sum_{i=1}^{l_0} 2\pi - \sum_{j,\mu} \int_{\Gamma_j^{\mu}} \varkappa_g ds$$

$$= 2\pi\chi(\Delta) - \sum_{i=1}^{n} (\pi - \alpha_i) - \sum_{i=1}^{l_0} \Omega(p_i) - \sum_{j=1}^{l_1} \Omega(L_j) - \int_{\Gamma} \varkappa_g ds.$$

But

$$\sum_{i=1}^{l_2} \Omega(\Delta_i) + \sum_{j=1}^{l_1} \Omega(L_j) + \sum_{k=1}^{l_1} \Omega(p_k) = \Omega(\Delta)$$

so that

$$\Omega(\Delta) = 2\pi\chi(\Delta) - \sum_{i=1}^{n} (\pi - \alpha_i) - \int_{\Gamma} \varkappa_g ds\,.$$

Stavroulakis proved in [3] and [4] that, if Δ is a simple connected sheet of the class C^2 in every point with except of m vertical points $p_1, \ldots, p_m$, $\Gamma = \mathrm{Fr}\,\Delta$ is a curve with n angled points, φ_j $(j = 1, \ldots, m)$ is a total angle around p_j, and the cones tangent to Δ in p_j are of the class C^2 in every point with the exception of the vertex p_j, then

$$(7) \qquad (n + 2m - 2)\pi - \sum_{j=1}^{n} \alpha_j + \int_{\Gamma} \varkappa_g ds + \iint_{\Delta} K\,dS = \sum_{j=1}^{m} \varphi_j\,.$$

Stavroulakis showed that this formula is equally true when the cone tangent in every vertical point has a simple closed characteristic curve [3] (i. e. when S is of II kind), or when all demitangents in every vertical point are identical (i. e., when S is of I kind). Then the corresponding angle φ in the formula (7) is zero [4].

We shall try to generalize this formula for the surfaces with edges and with vertical points of III kind. Without loss of generality we can consider a sheet Δ with the only one vertical point, for, if Δ is a sheet with a finite number of vertical points, then we can divide it into a finite number of sheets such that each of them contains only one vertical point.

Let Δ be the sheet with only one vertical point p_0. Then the formula (7) may be expressed as follows:

$$\iint_{\Delta} K\,dS + (2\pi - \varphi) = 2\pi - \sum_{j} (\pi - \alpha_j) - \int_{\Gamma} \varkappa_g\,ds\,. \tag{7'}$$

Let $\Delta = \Delta' \cup \Delta'' \cup \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \cup \Gamma_2 = \overline{\Delta'} \cap \overline{\Delta''}$ is an arc containing the vertical point p_0 of I or II kind with the total angle φ around it. From Theorem 1 and from the definition of $\Omega(\Delta)$ it follows that

$$\Omega(\Delta') + \Omega(\Delta'') + (2\pi - \varphi) = 2\pi - \sum_{j} (\pi - \alpha_j) - \int_{\mathrm{Fr}\,\Delta} \varkappa_g\,ds\,.$$

From this and from (7') it follows that

$$\Omega(\Delta') + \Omega(\Delta'') = \iint_{\Delta'} K\,dS + \iint_{\Delta''} K\,dS\,. \tag{8}$$

We shall prove the following

THEOREM 2. *Let Δ'' be a regular sheet such that* $\mathrm{Fr}(\Delta'')$ *contains at most a finite number of vertical points. Then*

$$\Omega(\Delta'') = \iint_{\Delta''} K\,dS\,.$$

(It is sufficient to prove Theorem 2 for a sheet with one vertical point p_0).

LEMMA. *Theorem 2 is true provided that the arcs Γ_1 and Γ_2 ($\Gamma_1 \cap \Gamma_2 = p$) are the geodesic lines of Δ ($\Delta = \Delta' \cup \Delta''$ as in* (8)).

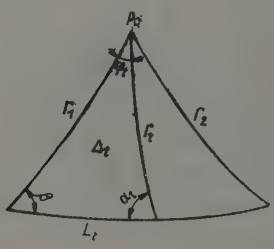

Fig. 1

Proof of the Lemma. According to (8), we have: $\Omega(\Delta') + \Omega(\Delta'') = \iint_{\Delta'} K\,dS + \iint_{\Delta''} K\,dS$ (and $\overline{\Delta' \cap \Delta''} = \overline{\Gamma_1 \cup \Gamma_2}$). We can find a parametric representation $p(\varrho, t)$ of a surface $\Delta - p_0$ such that the parametric lines Γ_t issuing from p_0: $t = \text{const.}$, are geodesic lines of $\Delta - p_0$. Let $\Gamma_t \subset \Delta_1$ and Δ_t be the subset of Δ_1 bounded by the arcs Γ_1, Γ_t and L_t ($L_t \subset \mathrm{Fr}\,\Delta$), let $\varphi_t = \sphericalangle(\Gamma_1, \Gamma_t)$ on Δ, $\alpha_t = \sphericalangle(L_t, \Gamma_t)$ on Δ, $\beta = \sphericalangle(L_t, \Gamma_1)$ on Δ and let $\lim_{t \to 1} \Gamma_t = \Gamma_1$.

We construct a retraction $F(t, p)$ of the set $\Delta' \cup \Delta'' \cup \Gamma_1$ into $\Delta'' \cup \Gamma_1$ ($\Delta \to \Gamma_1$) by putting

$F(t, p) = p$, if $p \in \Delta'' \cup \Gamma_1 \cup \Delta_t$,

$F(t, p) = q$, if $p \in \Delta' - \Delta_t$, where q is a point of Γ_t such that $\varrho_\Delta(q, p_0) = \varrho_\Delta(p, p_0)$.

Then

$$\lim_{t \to 1} \varphi_t = 0\,, \quad \lim_{t \to 1} (\pi - \alpha_t) = \beta\,, \quad \lim_{t \to 1} \int_{L_t} \varkappa_g(s)\,ds = 0\,. \tag{9}$$

(because, if $t' \leqslant t''$, we have $L_{t'} \supset L_{t''}$ and $\lim_{t \to 1} |L_t| = 0$).

It follows from the definition of Ω that

$$\Omega(\Delta_t) = 2\pi - (\pi - \alpha_t) - (\pi - \beta) - (\pi - \varphi_t) - \int_{L_t} \varkappa_g(s)\, ds\,.$$

According to (9),

$$\lim_{t\to 1} \Omega(\Delta_t) = 2\pi - \beta - \pi + \beta - \pi = 0\,,$$

hence

$$\lim_{t\to 1} \big(\Omega(\Delta') + \Omega(\Delta'')\big) = \Omega(\Delta'')\,. \tag{10}$$

The Gauss curvature on Δ is bounded [3], hence

$$\lim_{t\to 1} \iint_{\Delta'} K\, dS = 0$$

and therefore

$$\iint_{\Delta''} K\, dS = \Omega(\Delta'')\,.$$

Proof of Theorem 2. Now let Γ_1 be an arbitrary regular arc joining p_0 with $\mathrm{Fr}\,\Delta$. Let us take a sequence of geodesic broken lines $\{G_n\}$ on Δ, lying on one side of Γ_1 such that each of them has the ends as the only common points with Γ_1 and that $\sphericalangle(\Gamma_1, G_n) = \delta$ in the point of $\mathrm{Fr}\,\Delta$. Let $\sphericalangle(\Gamma_1, G_n) = \gamma$ in p_0 and let Δ_n be the sheet bounded by G_n and Γ_1. Then

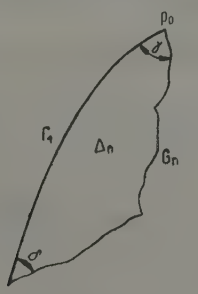

Fig. 2

$$\Omega(\Delta_n) = 2\pi - \sum_{i=1}^{m_n} (\pi - \alpha_i^n) - (\pi - \gamma) - (\pi - \delta) - \int_{\Gamma_1} \varkappa_g\, ds\,.$$

If $n\to\infty$, $G_n \to \Gamma_1$, then

$$\lim_{n\to\infty} \Omega(\Delta_n) = 2\pi + \int_{\Gamma_1} \varkappa_g\, ds - \pi - \pi - \int_{\Gamma_1} \varkappa_g\, ds = 0$$

for $[\gamma + \delta - \sum (\pi - \alpha_i^n)] \to \int_{\Gamma_1} \varkappa_g\, ds$ ([1], pp. 355-356).

Theorem 2 follows from the last equality and from the Lemma.

Corollary. *The Gauss-Bonnet formula is true provided that the regular sheet Δ has a finite number of vertical points of I or II kind on* $\mathrm{Fr}\,\Delta$.

This follows from Theorems 1 and 2.

THEOREM 3. *If a simple connected sheet Δ has a finite number of vertical points p_j $(j=1,\ldots,m)$ of* I, II *or* III *kind and a finite number of edges L_i^j $(i=1,\ldots,k_j,\ j=1,\ldots,m)$ issuing from p_j, then*

$$(11)\quad \iint_{\Delta} K\,dS = \Big(2\pi - \sum_j (\pi - \alpha_j) - \int_{\mathrm{Fr}\Delta} \varkappa_g\,ds\Big) - \sum_{i=1}^{m}(2\pi-\varphi_i) - \sum_{ij}\int_{L_i^j}(\varkappa_g^1+\varkappa_g^2)\,ds.$$

Proof. I. Firstly we shall prove Theorem 3 for the case of the sheet with the vertical points of I or II kind only. We can represent the sheet Δ as a union:

$$\Delta = \Delta_1 \cup \ldots \cup \Delta_k \cup L_1 \cup \ldots \cup L_k \cup p_0, \qquad \text{where}$$
$$\overline{\Delta}_i \cap \overline{\Delta}_{i+1} = L_i, \qquad i=1,\ldots,k-1;\ \overline{\Delta}_k \cap \overline{\Delta}_1 = L_k.$$

From Theorem 2 we have:

$$\Omega(\Delta_j) = \iint_{\Delta_j} K\,dS \qquad (j=1,\ldots,k),$$

hence

$$\sum_{j=1}^{k}\Omega(\Delta_j) = \iint_{\Delta} K\,dS.$$

It follows from this relation and from Theorem 1 that

$$\iint_{\Delta} K\,dS + \sum_{j=1}^{m}\Omega(p_j) + \sum_{ij}\Omega(L_i^j) = 2\pi - \sum_j(\pi-\alpha_j) - \int_{\mathrm{Fr}\Delta}\varkappa_g\,ds.$$

But

$$\Omega(p_j) = 2\pi - \varphi_j, \qquad \text{and} \qquad \Omega(L_i^j) = \int_{L_i^j}(\varkappa_g^1+\varkappa_g^2)\,ds.$$

Hence,

$$\iint_{\Delta} K\,dS = \Big(2\pi - \sum_j(\pi-\alpha_j) - \int_{\mathrm{Fr}\Delta}\varkappa_g(s)\,ds\Big) - \sum_{i=1}^{m}(2\pi-\varphi_i) - \sum_{ij}\int_{L_i^j}(\varkappa_g^1+\varkappa_g^2)\,ds.$$

II. Let now $p_0 \in \Delta$ be the vertical point of III kind. We can represent Δ in the form:

$$\Delta = \Delta_1 \cup \ldots \cup \Delta_k \cup L_1 \cup \ldots \cup L_k \cup p_0,$$

where

1°. $\overline{\Delta}_i \cap \overline{\Delta}_{i+1} = L_i$, $i=1,\ldots,k-1$; $\overline{\Delta}_k \cap \overline{\Delta}_1 = L_k$

and

2°. the part of the cone tangent in $p_0 \in \coprod_i \mathrm{Fr}\,\Delta_i$, which corresponds to Δ_j, is of I or II kind for each j.

Applying Corollary to each of the sheets Δ_i and proceeding like in the first part of this proof we obtain the formula (11). Hence, Theorem 3 is proved.

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MATHEMATICS

Independence in Some Abstract Algebras

by

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This paper contains some results concerning the general algebraic notion of independence, defined in [4] and [5] (see also [6] and [7]). The notation and definitions established in these papers are adopted here.

I investigate a certain class of abstract algebras, namely v-algebras defined below in § 2. They present a natural and simple generalization of abstract vector spaces. In particular, the general notion of independence in the domain of v-algebras possesses the principal properties of the linear independence (see § 5 and 6).

1. Dependence on variables. Distinguishability.

Let A be a non void set. We consider functions of many variables running over A. They may, in particular, be propositional functions, i. e. mappings of $A \times A \times \ldots \times A$ into a two-element set. We say that a function f of n variables ($n = 1, 2, \ldots$) *depends on the first variable*, whenever there are in A elements $a_1, a_2, \ldots, a_n$, and a_1', such that $f(a_1, a_2, \ldots, a_n) \neq f(a_1', a_2, \ldots, a_n)$.

(i) *A function not depending on any variable is constant.*

(ii) *A function f of n variables depends on the first variable if and only if the condition (i. e. the propositional function) $f(x_1, \ldots, x_n) = x_{n+1}$ depends on the first variable.*

(iii) *A condition which depends on no variable is fulfilled either always or never.*

Two functions of n variables are *distinguishable by the first variable*, whenever the equation $f = g$ depends on the first variable or, in other words, if and only if there are $a_1, \ldots, a_n$ and a_1' such that

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n) \quad \text{and} \quad f(a_1', a_2, \ldots, a_n) \neq g(a_1', a_2, \ldots, a_n)$$

(iv) *If two functions are not distinguishable by any variable, then they are either identical, or everywhere different.*

Let $(A, \boldsymbol{F})$ be an algebra.

(v) *If an algebraic operation* $f \in \boldsymbol{A}^{(n)}$ *(where* $n = 1, 2, \ldots$*) does not depend on the first variable, then there exists a* $g \in \boldsymbol{A}^{(n-1)}$ *such that identically* $f(x_1, x_2, \ldots, x_n) = g(x_2, \ldots, x_n)$.

Proof. Since f does not depend on x_1, the required equation is satisfied if we put $g(x_2, x_3, \ldots, x_n) = f(x_2, x_2, x_3, \ldots, x_n)$.

In view of the proposition 1 (iii) of [4], we have: $g \in \boldsymbol{A}^{(n-1)}$.

2. Definition and examples.

We say that an algebra $(A, \boldsymbol{F})$ is a *v-algebra*, when for each pair of algebraic operations: $f, g \in \boldsymbol{A}^{(n)}$ $(n = 1, 2, \ldots)$ distinguishable by the first variable there exists an algebraic operation $h \in \boldsymbol{A}^{(n-1)}$ such that the equation $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ is equivalent to the equation $x_1 = h(x_2, \ldots, x_n)$.

(Let us remember that $\boldsymbol{A}^{(0)}$ denote the set of all algebraic constants, i. e. the set of all values admitted by constant algebraic operations).

Example 1. Every vector space V over a field K is an algebra $(V, \boldsymbol{F})$; the class $\boldsymbol{F}$ of fundamental operations consists here of the addition of vectors and of all operations $f_a(x) = a \cdot x$ with a in K. The independence in the algebra $(V, \boldsymbol{F})$ is identical with the linear independence in the considered vector space (cf. [4], 4). The class of all algebraic operations in $(V, \boldsymbol{F})$ consists obviously of all functions of the form $a_1x_1 + \ldots + a_nx_n$ (a_j in K, x_j in V). It is easy to see that two such operations are distinguishable by the first variable if and only if their first coefficients are different. It follows therefore that $(V, \boldsymbol{F})$ is a v-algebra.

Example 2. Let W be a subspace of a vector space V. Denote by $(V, \boldsymbol{L})$ the algebra in which the class $\boldsymbol{L}$ of fundamental operations consists of the same operations as in Example 1 and all constant W-valued functions defined over V. It is easy to see that $(V, \boldsymbol{L})$ is a v-algebra.

For other examples and a general representation theorem for v-algebras see [8].

3. Direct conclusions.

From now on we suppose that $(A, \boldsymbol{F})$ *is a v-algebra.*

The definition of v-algebras and the propositions 1 (ii) and 1 (iv) of [4] imply

(i) *If* $f \in \boldsymbol{A}^{(m)}$, $g \in \boldsymbol{A}^{(n)}$, *the sequence* $1, 2, \ldots, r$ *is decomposed in two (not necessarily disjoint) sequences* $j_1, \ldots, j_m$ *and* $k_1, \ldots, k_n$, *and if the equation* $f(x_{j_1}, \ldots, x_{j_m}) = g(x_{k_1}, \ldots, x_{k_n})$ *depends on the variable* x_1, *then there exists an* $h \in \boldsymbol{A}^{(r-1)}$ *such that the preceding equation is equivalent to the following one:* $x_1 = h(x_2, \ldots, x_r)$.

Hence,

(ii) *If* $f \in A^{(n)}$ $(n = 1, 2, \ldots)$ *and* f *depends on the first variable, then there exists a* $g \in A^{(n)}$ *such that the equations* $x_{n+1} = f(x_1, \ldots, x_n)$ *and* $x_1 = g(x_2, \ldots, x_{n+1})$ *are equivalent.*

In fact, if f depends on x_1, then so does the equation $x_{n+1} = f(x_1, \ldots, x_n)$.

(iii) *If* f, $g \in A^{(n)}$ $(n = 2, 3, \ldots)$, f *and* g *are distinguishable by the first and not distinguishable by the* n*-th variable, then there exists an* $h \in A^{(n+2)}$ *such that the equation*

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \tag{1}$$

is equivalent to

$$x_1 = h(x_2, \ldots, x_{n-1}) . \tag{2}$$

The definition of v-algebras implies the existence of $h^* \in A^{(n-1)}$ such that (1) is equivalent to

$$x_1 = h^*(x_2, \ldots, x_n) . \tag{3}$$

Since f and g are not distinguishable by x_n, the Eq. (1) and consequently the Eq. (3) do not depend on x_n. Therefore, in view of 1(ii), h^* does not depend on x_n. By 1(v) there is an $h \in A^{(n-2)}$ such that (2) and (3) are equivalent.

(iv) *If* $f, g \in A^{(n)}$ $(n = 1, 2, \ldots)$ *are distinguishable by the first variable, then, for each* $b_1, b_1', b_2, \ldots, b_n$, *if* $b_1' \neq b_1$ *and* $f(b_1, b_2, \ldots, b_n) = g(b_1, b_2, \ldots, b_n)$ *then* $f(b_1', b_2, \ldots, b_n) \neq g(b_1', b_2, \ldots, b_n)$.

By the definition of v-algebras, the Eq. (1) is equivalent to an equation $x_1 = h(x_2, \ldots, x_n)$, where $h \in A^{(n-1)}$.

Since (1) holds for $x_j = b_j$ $(j = 1, 2, \ldots, n)$, we have $h(b_2, \ldots, b_n) = b_1 \neq b_1'$ and consequently $f(b_1', b_2, \ldots, b_n) \neq g(b_1', b_2, \ldots, b_n)$.

4. Subalgebras. Algebras of algebraic operations.

(i) *Any subalgebra* B *of a* v*-algebra* $(A, \boldsymbol{F})$ *is also a* v*-algebra. Moreover, if* B *contains at least two elements, then algebraic operations which are different in* A *are also different in* B.

If two algebraic operations of n variables f and g are distinguishable by the first variable in B, then they are distinguishable in A. Since $(A, \boldsymbol{F})$ is a v-algebra, there exists in A an algebraic operation h of $n-1$ variables such that the equation $f = g$ is equivalent to $x_1 = h(x_2, \ldots, x_n)$ in A, and thus in B.

The second part of the theorem follows from 1(i) and 3(iv).

(ii) *The algebra* $(A^{(m)}, \boldsymbol{F})$ *of all algebraic operations of* m *variables in a* v*-algebra* $(A, \boldsymbol{F})$ *is also a* v*-algebra.*

Let $f, g \in A^{(m)}$ be distinguishable by the first variable, when they are treated as operations in $(A^{(n)}, F)$. In other words, there exist in $A^{(m)}$ operations $f_1, \ldots, f_m$ and f_1' such that $f(f_1, f_2, \ldots, f_m) = g(f_1, f_2, \ldots, f_m)$ and $f(f_1', f_2, \ldots, f_m) = g(f_1', f_2, \ldots, f_m)$.

Thus, there exist elements $b_1, \ldots, b_n$ in A such that putting $a_1' = f_1'(b_1, \ldots, b_n)$ and $a_j = f_j(b_1, \ldots, b_n)$ for $j = 1, 2, \ldots, m$ we have $f(a_1', a_2, \ldots, a_n) \neq g(a_1', a_2, \ldots, a_n)$ and $f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n)$.

Thus, f and g, treated as algebraic operations in the v-algebra (A, F), are distinguishable by the first variable. Thus, there is an h in $A^{(n-1)}$ such that the equations

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \quad \text{and} \quad x_1 = h(x_2, \ldots, x_n)$$

are equivalent for x_j in A. It is now easy to infer that the equations

$$f(f_1, \ldots, f_n) = g(f_1, \ldots, f_n) \quad \text{and} \quad f_1 = h(f_2, \ldots, f_n)$$

are equivalent for f_j in $A^{(m)}$.

5. Independent elements and generators.

The first four theorems proved below are converse to theorems valid for all algebras (cf. e. g. [4], theorem 2 (iii)).

(i) *Any element of A which is not an algebraic constant (i. e. belonging to $A \setminus A^{(0)}$) forms a set of independent elements).*

In fact if a unit set $\{c\}$ is not a set of independent elements, then there exist two different operations $f, g \in A^{(1)}$ such that $f(c) = g(c)$. Then, by the definition of v-algebras, there exists an algebraic constant d such that $f(x) = g(x)$ if and only if $x = d$. Consequently, $c = d$.

(ii) *If I is a set of independent elements in a subalgebra B of A, then I is a set of independent elements in A.*

This is a consequence of the second part of 4(i).

(iii) *If $a_1, \ldots, a_n, a_{n+1}$ $(n = 1, 2, \ldots)$ are dependent elements of A while $a_1, \ldots, a_n$ are independent, then there exists an h in $A^{(n)}$ such that $a_{n+1} = h(a_1, \ldots, a_n)$.*

In fact, there exist, by the first hypothesis, two different operations $f, g \in A^{(n+1)}$ such that

$$f(a_1, \ldots, a_n, a_{n+1}) = g(a_1, \ldots, a_n, a_{n+1}).$$

f and g are distinguishable by the $(n+1)$-th variable, since, otherwise, it would follow from 1(iv) and 3(iii) that $a_1, \ldots, a_n$ are dependent, which would contradict the second hypothesis.

From (iii) and (i) we obtain

(iv) *If the set J contained in A and containing at least two elements is not generated by any of its proper subsets, then J is a set of independent elements.*

Theorem (iii) implies, by an argument familiar in the theory of vector spaces (see e. g. [1], p. 178-9, Theorem 7):

(v) *If there exists a set of n generators of (A, F) $(n = 1, 2, \ldots)$, then each set of independent elements contains at most n elements.*

Now we prove a proposition which was formulated by Whitney [9] as the axiom of linear independence. Consequently, v-algebras with sets of independent elements are "Abhängigkeitsräume" in the sense formulated in [2].

(vi) *If N is a set of n independent elements $(n = 1, 2, \ldots)$ and M a set of $n+1$ independent elements, then there is an a in $M \setminus N$ such that $N \cup \{a\}$ is a set of independent elements.*

Let us suppose the contrary. Thus, by (iii), M is contained in the subalegbra $\bar{N}$ generated by N. Since M is a set of independent elements in A, it is *a fortiori* a set of independent elements in $\bar{N}$. By 4(i) N is a v-algebra which leads to a contradiction with (v) since $\bar{N}$ possesses n generators and $n+1$ independent elements.

6. Bases

In each algebra any basis (i. e. any set of independent generators) is at the same time a minimal set of generators and a maximal set of independent elements. In the domain of v-algebras some converse relations occur.

If $A = A^{(0)}$, i. e. if the algebra A consists only of its algebraic constants, there are no independent elements, no basis, and every one-element set is a minimal set of generators. Rejecting this exceptional case, we obtain the following theorem for v-algebras:

(i) *If $A \neq A^{(0)}$, then following statements are equivalent*: (a) *B is a basis of A*, (b) *B is a minimal set of generators of A*, (c) *B is a maximal set of independent elements in A.*

The implication (b)$\rightarrow$(a) follows from 5(iv) in the case when B contains at least two elements. If B contains only one element, it does not belong to the subalgebra $A^{(0)}$ and, by 5(i), B is a set of independent elements.

The implication (c)$\rightarrow$(a) follows from the following sharper proposition:

(ii) *If G is a set of generators of (A, F) and B is its non void subset of independent elements, maximal in G, then B is a basis of A.*

For each $d \in G$ there exists obviously a sequence $b_1, \ldots, b_n$ of elements of B such that the elements $b_1, \ldots, b_n, d$ are dependent. Consequently, by 5(iii), d belongs to the subalgebra $\bar{B}$ generated by B. Therefore, $\bar{B}$ contains G, whence B is a set of generators of A.

Concerning the problem of existence of a basis let us first remark that, by Kuratowski-Zorn lemma, in any subset D of any algebra and for any non void set E of independent elements of D there exists a set I of independent elements, maximal in D and such that $E \subset I \subset D$.

Hence, in view of (ii), we have for v-algebras:

(iii) *If I is a non void set of independent elements and G a set of generators of (A, F), then there exists a basis B of this algebra, such that $I \subset B \subset I \cup G$.*

Consequently, by 5(i), we obtain

(iv) *If $A \neq A^{(0)}$, then there exists a basis.*

(v) *Any two bases have the same cardinal number.*

On account of 5(v), if one basis is finite, then any other basis consists of the same number of elements. Let us add that this implication valid for v-algebras is false for arbitrary abstract algebras (see [3]). Nevertheless, it is easy to see that if an arbitrary abstract algebra has an infinite basis, then each of its bases has the same cardinal number.

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MATHEMATICS

Representation Theorem for Marczewski's Algebras

by

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Let us consider an arbitrary non-empty set A. Let $\boldsymbol{A}$ be a class of A-valued functions of finitely many variables running over A such that 1^0 the functions defined by the formula $f(x_1, \ldots, x_n) = x_k$ $(n = 1, 2, \ldots, k = 1, 2, \ldots, n)$ belong to $\boldsymbol{A}$; 2^0 $\boldsymbol{A}$ is closed with respect to the superposition of functions. The system $(A, \boldsymbol{A})$ will be called an *algebra*.

By $\boldsymbol{A}^{(0)}$ we shall denote the class of all constants belonging to $\boldsymbol{A}$. Further, by $\boldsymbol{A}^{(n)}$ $(n \geqslant 1)$ we shall denote the class of all functions of n variables belonging to $\boldsymbol{A}$. If $1 \leqslant k \leqslant n$, then $\boldsymbol{A}^{(n,k)}$ will denote the subclass of $\boldsymbol{A}^{(n)}$ containing all functions depending on at most k variables, i. e. $f \in \boldsymbol{A}^{(n,k)}$ if there is a function $g \in \boldsymbol{A}^{(k)}$ such that $f(x_1, \ldots, x_n) = g(x_{i_1}, \ldots, x_{i_k})$ for a system of indices $i_1, \ldots, i_k$ and for every $x_1, \ldots, x_n \in A$.

Let $f, g \in \boldsymbol{A}^{(n)}$ $(n \geqslant 1)$. We say that the equality

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$$

depends on the variable x_j $(1 \leqslant j \leqslant n)$ if there exists a system $a_1, \ldots, a_n, a'_j$ of elements belonging to A for which

$$f(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n) = g(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)$$

and

$$f(a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n) \neq g(a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n).$$

An algebra $(A, \boldsymbol{A})$ is called a *Marczewski's algebra* if for every pair of integers j, n $(1 \leqslant j \leqslant n)$ and for every pair of functions $f, g \in \boldsymbol{A}^{(n)}$ for which the equality

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \tag{1}$$

depends on x_j, there exists a function $h \in \boldsymbol{A}^{(n-1)}$ such that the equality (1) is equivalent to the following one

$$x_j = h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

The study of these algebras was initiated by E. Marczewski [1] (called by him v-algebras).

Now we shall give some examples of Marczewski's algebras.

1. Let A be a linear space over a field K and let A_0 be a linear subspace of A. If $\boldsymbol{A}$ is the class of all functions f defined as follows

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \ldots, \lambda_n \in K$ and $a \in A_0$, then $(A, \boldsymbol{A})$ is a Marczewski's algebra. In this case we have the relations

$$\boldsymbol{A}^{(0)} \neq 0, \quad \boldsymbol{A}^{(n)} \neq \boldsymbol{A}^{(n,1)} \quad \text{for } n \geqslant 2.$$

This example is due to Marczewski.

2. Let A be a linear space over a field K and let A_0 be a linear subspace of A. If $\boldsymbol{A}$ is the class of all functions f defined as follows

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \ldots, \lambda_n \in K$, $\sum_{k=1}^{n} \lambda_k = 1$ and $a \in A_0$, then $(A, \boldsymbol{A})$ is a Marczewski's algebra. In this case we have the relations

$$\boldsymbol{A}^{(0)} = 0, \quad \boldsymbol{A}^{(n)} \neq \boldsymbol{A}^{(n,1)} \quad \text{for } n \geqslant 3.$$

(Let us remark that $\boldsymbol{A}^{(2)} = \boldsymbol{A}^{(2,1)}$ in the case, when the field K contains two elements only).

3. Let $\boldsymbol{G}$ be a group of transformations of a non-empty set A. We suppose that every non identical transformation has at most one fixed point in A.

We say that a subset $B \subset A$ is *normal* with respect to the group $\boldsymbol{G}$, if B contains fixed points of all non identical transformations from $\boldsymbol{G}$ and $g(B) \subset B$ for every $g \in \boldsymbol{G}$. We remark that if a transformation from $\boldsymbol{G}$ has no fixed points, then the empty set is normal with respect to $\boldsymbol{G}$.

Let A_0 be a subset of A normal with respect to $\boldsymbol{G}$. If $\boldsymbol{A}$ is the class of all functions f defined as follows

$$f(x_1, \ldots, x_n) = g(x_j) \quad (1 \leqslant j \leqslant n)$$

or

$$f(x_1, \ldots, x_n) = a,$$

where $g \in \boldsymbol{G}$ and $a \in A_0$, then $(A, \boldsymbol{A})$ is a Marczewski's algebra.

In this case we have the relation

$$\boldsymbol{A}^{(n)} = \boldsymbol{A}^{(n,1)}, \quad n \geqslant 1.$$

The following representation theorem gives an answer to a problem raised by E. Marczewski.

THEOREM. *Let* $(A, \boldsymbol{A})$ *be a Marczewski's algebra.*

(i) *If* $\boldsymbol{A}^{(n)} \neq 0$ *and* $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, *then there is a field* K *such that* A *is linear space over* K. *Furthermore, there exists a linear subspace* A_0 *of* A *such that* $\boldsymbol{A}$ *is the class of all functions* f *defined as follows*

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k ,$$

where $\lambda_1, \ldots, \lambda_n \in K$ *and* $a \in A_0$.

(ii) *If* $\boldsymbol{A}^{(0)} = 0$ *and* $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, *then there is a field* K *such that* A *is a linear space over* K. *Further, there exists a linear subspace* A_0 *of* A *such that* A *is the class of all functions* f *defined as follows*

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a ,$$

where $\lambda_1, \ldots, \lambda_n \in K$, $\sum_{k=1}^{n} \lambda_k = 1$, *and* $a \in A_0$.

(iii) *If* $\boldsymbol{A}^{(3)} = \boldsymbol{A}^{(3,1)}$, *then there is a group* $\boldsymbol{G}$ *of transformations of the set* A, *such that every non identical transformation has at most one fixed point. Moreover, there is a subset* $A_0 \subset A$ *normal with respect to the group* $\boldsymbol{G}$ *such that* $\boldsymbol{A}$ *is the class of all functions* f *defined as follows*

$$f(x_1, \ldots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n) ,$$

or

$$f(x_1, \ldots, x_n) = a ,$$

where $g \in \boldsymbol{G}$ *and* $a \in A_0$.

The proof will be published in the Fundamenta Mathematicae.

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MATHEMATICS

Analysis of the Cantor-Bendixson Theorem by Means of the Analytic Hierarchy

by

G. KREISEL

Presented by A. MOSTOWSKI on August 25, 1959

We consider sets of points on the segment $[0, 1]$. Closed sets are denoted by F (or F_i); they are determined by sets G (or G_i) of open intervals with rational end points whose union is the complement of $F(F_i)$. — "Open" and "closed" are understood relative to $[0, 1]$. — F is said to be in Π^1_n or Σ^1_n (in the notation of [5], pp. 180-181) if there is a corresponding set *) G of rational intervals (or, natural numbers under one of the standard enumerations of such intervals) which is in Π^1_n or Σ^1_n.

One form of the theorem of Cantor-Bendixson (CB) states:

Any closed set is the union of a perfect set and an enumerable set.

Since the partition is unique, we may denote the perfect part, or kernel, of F by F_p, and the rest by F_s ("s" for scattered). The literature contains two main proofs of CB. A short one which identifies F_p with the set of condensation points of F, and a more detailed one which shows (i) that the transfinite sequence of derived sets $F^{(\eta)}$ is constant for $\eta \geqslant \eta_0$, $\eta_0 < \Omega$, and (ii) that $F_p = F^{(\eta_0)}$. The difference between these proofs is sometimes seen in the use of the axiom of choice in the short proof, e. g. [3]. Kleene's analytic hierarchy suggests a different emphasis:

If F is in Π^1_1, the short proof shows only that F_p is in Σ^1_2 while the other shows that F_p is in Π^1_1 too and F_s can be enumerated by a function recursive in Π^1_1.

The first part is obvious (Theorem 0); the second, which, in a different terminology, was given by Lorenzen [4], is proved in Theorem 1, and the third in Theorem 2. As a corollary it follows that, for F in Π^1_1, $F^{(\eta} = F^{(\eta_0)}$ for some η_0 not exceeding the first non-recursive ordinal (ω_1).

*) Lemma A below shows that in the cases which interest us most, the set of *all* open rational intervals complementary to F belongs to the class if any G belongs to the class.

Theorems 1 *and* 2 *are essentially optimal.* For, in Theorem 3, we construct a recursively closed set F^*, i. e. F^* in $\Pi_0^0 \cap \Sigma_0^0$, such that neither F_p^* nor F_s^* is hyperarithmetic. This implies that neither is in Σ_1^1 (where we say that F_s is in Σ_1^1 if it can be enumerated by a function whose graph is in Σ_1^1). It follows also that $\min \eta_0^* = \omega_1$.

Discussion. The results obtained here apply not only to the full segment $[0, 1]$, but also to its intersection with various classes $\boldsymbol{K}$ (of real numbers) associated with classes K of number theoretic functions (Mostowski [5], p. 181), particularly for $K = \Pi_\alpha^0 \cap \Sigma_\alpha^0$, where α is a recursive limit *) ordinal, and $K = \Pi_1^1 \cap \Sigma_1^1$. It follows from Lemma A that, when relativized to any of these classes (and for the full segment), CB is equivalent to a formula $(\sigma)(E\tau)(\xi)R(\sigma, \tau, \xi)$, where σ, τ, ξ range over the (relevant) class of real numbers and R is arithmetic in F. Analysis of Theorem 3 shows that CB is false when relativized to the $\boldsymbol{K}$ mentioned above, with the same *) closed set F^* as counter example in all cases.

Theorem 3 was conjectured in [2], and announced in [2a] (with a formulation of the impredicative character of CB) at the winter meeting of the Association for Symbolic Logic, January 1959.

Operations on closed sets. Lemma A. *The following properties and sets are arithmetic in* F, *and the implied equivalences hold for any class of real numbers containing those which are arithmetic in* F. *For rational* a *and* b, $0 \leqslant a < b \leqslant 1$,

(1) $F \cap [a, b]$ *contains at least (at most)* n *points,*

(2) $F \cap [a, b]$ *contains at most a finite number of points,*

(3) $F \cap [a, b]$ *is perfect (not empty, no isolated points),*

(4) *the derived set* $F^{(1)}$ *of* F,

(5) *an enumeration of the isolated points of* F,

(6) *the set* G_F *of all rational intervals complementary to* F.

All these results are almost immediate consequences of the proof of the Heine-Borel theorem. Its standard proof shows that, if $[a, b]$ cannot be covered by any finite subset of G, i. e. $(a_i, b_i) \in G$, $1 \leqslant i \leqslant N$, $a_j < a_{j+1} < b_j < b_{j+1}$, $1 \leqslant j < N$, then $[a, b]$ contains a point which is arithmetic in F, and is a member of the set F. Thus, the Heine-Borel theorem holds for any class of real numbers which includes those that are arithmetic in F.

*) The use of limit ordinals recommends itself in any case because there the same class $\boldsymbol{K}$ is obtained for the different representations of real numbers by means of number theoretic functions.

**) We mean the same set of complementary intervals, the content of $F^* \cap \boldsymbol{K}$ is generally different for different $\boldsymbol{K}$. Note that, for $K \subset \Pi_1^1 \cap \Sigma_1^1$, $F_p \cap \boldsymbol{K}$ is empty. So, for such, K, F^* is essentially a counter example to the second half of CB which asserts that the scattered part of a closed set is enumerable (by a function of K).

(1) $F \cap [a, b]$ contains at least 1 point $\equiv [a, b]$ cannot be covered by any finite set of elements of G.

$F \cap [a, b]$ contains at least n points $\equiv$ there is a sequence of rationals $a = a_1 < b_1 < \ldots b_i < a_{i+1} < b_{i+1} < \ldots b_n = b$, such that each $F \cap [a_i, b_i]$, $1 \leqslant i \leqslant n$, contains at least 1 point.

At most $n \equiv$ not $n+1$.

(2) $F \cap [a, b]$ contains a finite number of points $\equiv$ for some n, $F \cap [a, b]$ contains at most n points. Apply (1).

(3) $F \cap [a, b]$ is perfect $\equiv F \cap [a, b]$ contains at least 1 point, and, for all rational c, d, $a \leqslant c < d \leqslant b$, if $F \cap [c, d]$ contains 1 point it contains at least 2 (no isolated points) *).

(4) (a, b) is complementary to $F^{(1)} \equiv F \cap [c, d]$ is finite for all rational c and d such that $a < c < d < b$. Apply (2).

(5) We enumerate the isolated points by a function arithmetic in F in the following (standard) manner. Choose some enumeration of all rational intervals $[r_i, s_i]$. By (1), we determine the least $i = i_0$, such that $F \cap [r_i, s_i]$ contains exactly one point, and, if $i_0, \ldots, i_m$, $m \leqslant n$, have been determined, i_{n+1} is the smallest $i > i_n$ (if there is one) such that (a) $[r_i, s_i]$ is disjoint from each $[r_j, s_j]$, $j = i_m$, $m \leqslant n$, and (b) $F \cap [r_i, s_i]$ contains exactly one point. This construction is arithmetic in F.

(6) (a, b) is complementary to $F \equiv F \cap (a, b)$ is empty $\equiv F \cap [c, d]$ is empty for all rational c and d, $a < c < d < b$. Apply (1).

To utilize (known) results on classes of number theoretic functions K for the associated classes of real numbers we need the familiar recursive mapping μ which takes the number theoretic function α into the real number

$$\sum_{n=0}^{\infty} 2^{-(n+1)-\sum_{i=0}^{n} \alpha(i)}, \qquad \alpha(i) \geqslant 0 .$$

This transformation maps the α one-one onto $(0, 1]$. Closed sets, are evidently not mapped onto closed sets. Lemma B below describes the differences which we use.

Definition. Let q denote the finite sequence $[a_0, \ldots, a_n]$ of natural numbers, $q_1 = 2^{-(a_0+1)} + \ldots + 2^{-(a_0+\ldots+a_n+n+1)}$, (q) denotes the open interval

$$(q_1, q_1 + 2^{-(a_0+a_1+\ldots+a_n+n+1)}) ,$$

$[a_0, \ldots, a_n]$ denotes the (elementary) neighbourhood of all number theoretic functions α, which satisfy $\alpha(i) = a_i$, $i \leqslant n$. A predicate $\Phi(q)$ uniquely determines the closed set F whose complement is the union of

$$\{[a_0, \ldots, a_n] | \Phi(q)\} .$$

*) This elegant formulation is due to Gödel.

LEMMA B. *If F is the complement (with respect to $[0,1]$) of the union of those (q) for which $\Phi(q)$ then* (i) $F \supset \mu(\boldsymbol{F})$, *and* (ii) $F - \mu(\boldsymbol{F})$ *is enumerable by a function recursive in the predicate $\Phi(q)$.*

$\mu([a_0, \ldots, a_n]) \supset (q)$ since $\mu([a_0, \ldots, a_n])$ consists just of (q) together with the right-hand end point of (q). Taking complements, we get $F \supset \mu(\boldsymbol{F})$.

Note that two elementary neighbourhoods (in the space of functions α) are either disjoint or one contains the other. Hence, if $\Phi_1(q)$ is defined to be

$$\Phi(q) \,\&\, (m)\big[m < n \to \neg\Phi([a_0, \ldots, a_m])\big],$$

hence $\hat{q}\Phi_1(q)$ is recursive in $\hat{q}\Phi(q)$ and $\boldsymbol{F}_1 = \boldsymbol{F}$. Then, $F - \mu(\boldsymbol{F})$ consists of the right-hand end points of those (q) for which $\Phi_1(q)$ holds.

COROLLARIES. *If $\boldsymbol{F}$ is not enumerable, neither is F. If $\boldsymbol{F}$ contains a function of degree $\boldsymbol{a}$, then F contains a real number of degree $\boldsymbol{a}$, since both μ and its inverse are recursive.*

THE THEOREM OF CANTOR-BENDIXSON. THEOREM 0. *If F is in Π^1_1, then F_p is in Σ^1_2.*

For $(a,b) \in G_p \equiv F \cap (a,b)$ is enumerable, i. e. there is a sequence ξ_n of real numbers such that for all real numbers η,

$$\eta \notin (a,b) \vee \eta \notin F \vee (En)(\eta = \xi_n).$$

If F is in Π^1_1, the function quantifier in $\eta \notin F$ can be absorbed, and so the set of (a,b) for which $F \cap (a,b)$ is enumerable, is in Σ^1_2.

THEOREM 1. (Lorenzen [4]). *If F is in Π^1_1, then F_p is in Π^1_1.*

We use the notation introduced at the beginning of this note.

Consider the least class G_0 of rational open intervals satisfying the two conditions:

(1) G_0 contains G (the set of complementary intervals of F);

(2) for all rational a, and b, if $F_0 \cap [a,b]$ is not empty, then $F_0 \cap [a,b]$ contains at least two points.

By Lemma A (3), these conditions are arithmetic. So G_0 is in Π^1_1 if F is in Π^1_1. F_0 is the maximal perfect set contained in F since G_0 is defined to be the *least* class satisfying (1) and (2), and so $F_0 = F_p$.

THEOREM 2. *If F is in Π^1_1, F_s can be enumerated by a function which is recursive in Π^1_1.*

We analyze the proof of Theorem 1 by means of recursive ordinals, as developed in [6]. We define a sequence G_e, for $e \in O$, such that $G_0 = \bigcup G_e$, $e \in O$. Given G_e, let H_e denote the set of complementary intervals of the derived set $F_e^{(1)}$. By Lemma A (4), H_e is arithmetic in G_e. $G_1 = G$ (the set of complementary intervals of F).

If $e = 2^f$, $G_e = H_f$.

If $e = 3.5^y$, $G_e = \bigcup_{n=0}^{\infty} G_{\{y\}(n)}$.

By [6], $G_0 = \bigcup G_e$, $e \in O$.

Now, by Lemma A (5), we have a uniform arithmetic method of enumerating (with repetition) the isolated points of F_e (if any), and their set theoretic sum is F_s. Thus, the enumeration is recursive in Π^1_1.

Note that Theorems 1 and 2 (and hence 0) also hold relative to arbitrary F, F_p being Π^1_1 in F and F_s enumerated by a function recursive in a set which is Π^1_1 in F.

In the proof of Theorem 3 below we shall use the following lemma, due to Spector [7], although our original proof depended on a weaker (related) result.

LEMMA. *There is a recursive $R(n, m)$ such that*

$$e \in O = (E!\,\alpha)(x)R[\bar{\alpha}(x), e] \equiv (E\alpha)_{HYP}(x)R[\bar{\alpha}(x), e],$$

and, moreover, for $e \in O$,

$$\iota\alpha(x)R[\bar{\alpha}(x), e] \text{ is of degree } |e|.$$

(To apply Lemma B in full, observe that we may choose R so that, for $x < y$,

$$R[\bar{\alpha}(x), e] \to R[\bar{\alpha}(y), e].)$$

THEOREM 3. *There is a recursively closed set F^* (in $\Pi^0_0 \cap \Sigma^0_0$) such that F^*_p is not hyperarithmetic (HYP) and F^*_s cannot be enumerated by a HYP function.*

Choose a recursive set of closed disjoint intervals J_e in $[0, 1]$, e. g. $J_e = [2^{-e} - 2^{-(e+2)}, 2^{-e}]$, and let a_e be the image of the subset a of $[0, 1]$ under the linear mapping of $[0, 1]$ into J_e, i. e., $\xi \to 2^{-e} - 2^{-(e+2)} + 2^{-(e+2)}\xi$. As in Lemma B, we consider the set of open intervals $(q)_e$, where q satisfies $\neg R(q, e)$, for $e = 0, 1, \ldots$ The complement of their union is the set F^*.

(a) For each $e \in O$, by Spector's lemma $(E!\,\alpha)(x)R(\bar{\alpha}(x), e)$, and so by the corollary to Lemma B, $F^* \cap J_e$ is enumerable, and therefore $\subseteq F^*_s$. Also by Lemma B, $F^* \cap J_e$ contains a real number of degree $|e|$. But the degrees of real numbers enumerated by a HYP function do not exceed the degree of the function itself, and so F^*_s is not enumerable by a HYP function.

(b) Recall that any isolated function which satisfies $(x)A[\bar{\alpha}(x)]$, is HYP. So, if

$$(E\alpha)(x)R[\bar{\alpha}(x), e] \;\&\; \neg\,(E\alpha)_{HYP}(x)R[\bar{\alpha}(x), e],$$

then $\hat{\alpha}(x)R[\bar{\alpha}(x), e]$ is not enumerable. By Lemma B, if I_e denotes the interior of J_e, $I_e \cap F^*_p$ is not empty. Thus $e \in O$ is equivalent to

$$(E\alpha)(x)R[\bar{\alpha}(x), e] \;\&\; (I_e \cap F^*_p \text{ is empty}).$$

By Lemma A (6), if F is HYP, so is the set of *all* its complementary intervals, and so $I_e \cap F_p^*$ would be HYP, if F_p^* were HYP. But in this case, by the above, $e \in O$ would be in Σ_1^1, which contradicts Kleene's result [1]. Hence, F_p^* is not HYP.

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ASTROPHYSICS

On Gas Expansion in the Central Parts of the Galaxy

by

S. GRZĘDZIELSKI

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An explanation of the sudden velocity decrease of expanding gas in the central parts of the Galaxy is suggested. The braking mechanism consists in a simple aerodynamic interaction between the neutral gas and the H II regions system merged in an axisymmetrical magnetic field. A transversal field of the order of 10^{-5} Gauss is sufficient to stop the expanding gas and to produce the observed H I density increase.

Several years ago, observations by Kwee, Muller and Westerhout [1] revealed the existence of long tails in the 21 cm line profiles in the direction of the Galactic Centre. These tails were at first believed to be due to a strong turbulence of neutral hydrogen. A more detailed analysis of the profiles performed at Dwingeloo by van Woerden, Rougoor and Oort [7] disclosed a trace of a spiral (or circular) arm at a distance of some 3 kpc from the centre, expanding at a velocity of 50 km/sec. Furthermore, in a recent paper by Westerhout [6] are mentioned Rougoor's unpublished results indicating "that not only this spiral arm, but also a large part, if not all, of the high-velocity neutral hydrogen in the centre region is expanding with velocities up to 200 km/sec".

The expanding gas participates probably in the general rotational motion. The transversal (rotational) velocity, obtained from the observed maximum radial velocity under the assumption of circular motion reveals a fast increase with growing distance r from the Galactic Centre; the real increase can be even more rapid, if we take into account the influence of the expansion velocities on the observed maxima. In consequence, the expanding hydrogen must intensively acquire the angular momentum.

Detailed investigations of continuous radiation at 1390 Mc/s completed by Westerhout [6] indicate a strong increase of H II regions number at $r = 2-4$ kpc, with a maximum at $r = 3.5-4.0$ kpc (see Fig. 1). In the same region, at $r = 4$ kpc, we also observe a sudden increase of neutral hydrogen density (Westerhout [5]). On the other hand, at $r = 3$ kpc

there occurs also a sudden decrease of expansion velocity from some 200 km/sec to some 50 km/sec. As Westerhout [6] suggested there must exist a physical connection between these phenomena, with magnetic and gravitational forces playing a determinant role.

In the present paper we are not considering the causes of gas expansion. We will limit our discussion to the dynamic behaviour of the

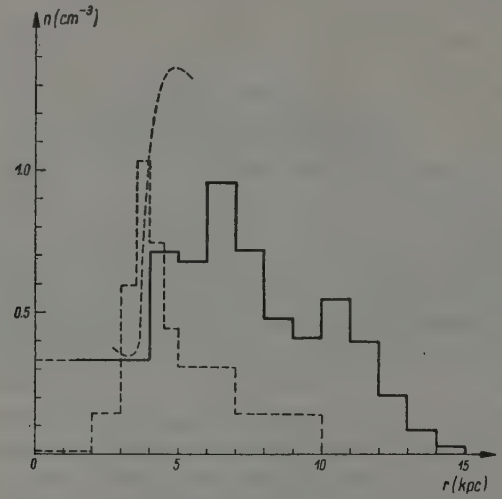

Fig. 1. Density *versus* the distance from the centre. Solid line — the observed neutral gas density (Westerhout [5]). Dashed line — the observed ionized gas density for $N = 10$ (Westerhout [6]). Dotted line — the computed density distribution of the neutral gas (present paper).

expanding gas in the region $r > 2.5$ kpc, supposing that the expansion causes are not operating there.

We consider a system of axial symmetry — the axis coinciding with the galactic axis, with the magnetic lines of force lying in the galactic plane and supported by external currents. The magnetic field is frozen into the ionized gas (H II regions system) so, that the individual regions can move along the lines of force only. The neutral hydrogen is not directly tied to the magnetic field. It can interact, however, with the field by means of H II regions and its own ionized substratum of very small density. We neglect the influence of this second agent as compared to the first one. To some extent it is a consequence of lack of the ionizing radiation density data. For the same reason we are not discussing the ionization and recombination processes, but considering the density distribution of the ionized gas as given.

As mentioned above, the expanding gas must acquire the angular momentum. The mean rate of this gain must be of the order of 2×10^{12} cgs

units per gram and second. Neither molecular viscosity nor turbulent momentum transfer can assure such a rate; a large scale magnetic field, however, with the radial component of the order of 3×10^{-6} Gauss seems to be quite sufficient. The existence of the radial component of relevant intensity is confirmed a posteriori.

Consequently, we can admit that the ionized gas participates in the rotational motion and that the centripetal gravitational force is cancelled by the centrifugal one. The neutral gas participates in the rotation too, owing to the interaction with the H II regions. A substantial retardment, however, cannot be excluded. The neglecting for neutral hydrogen of the centrifugal and gravitational as well as the ionized substratum interaction terms, means that we consider the lower limit of the braking mechanism.

For the sake of simplicity we suppose the problem to be stationary and independent of the z co-ordinate. Thus, in the polar co-ordinates, the equations of motion of the neutral and ionized gas have the forms

$$\varrho_1 v_r \frac{dv_r}{dr} = f_p + f_t + f_i, \tag{1}$$

$$\varrho_2 u_r \frac{du_r}{dr} = f_m - f_i, \tag{2}$$

where ϱ_1, v_r and ϱ_2, u_r are the mass densities and radial velocities of the neutral and ionized gas, respectively (ϱ_2 is a smoothed density equal nm_H in Westerhout notations [6]). The f's describe the neutral gas pressure (f_p), turbulent pressure (f_t), interaction of two media (f_i), and magnetic (f_m) force densities.

Moreover, we can obtain three independent integrals. One from the continuity equation:

$$r(\varrho_1 v_r + \varrho_2 u_r) = S = \text{const} \tag{3}$$

and two from the field equations:

$$u_r H_\varphi - u_\varphi H_r = \Lambda = \text{const} \tag{4}$$

(freezing condition) and

$$rH_r = \Gamma = \text{const} \tag{5}$$

(divergence condition).

Finally, we have the equations for the transversal velocities

$$v_\varphi = u_\varphi = (rK_\omega^{\sim})^{1/2}, \tag{6}$$

where $K_\omega^{\sim}$ is the gravitational acceleration, H_r and H_φ are the radial and transversal components of the magnetic field; v_φ and u_φ are the transversal velocities of the neutral and ionized gas, respectively.

Now, we assume the neutral gas to be isothermal, so

$$f_p = -c^2 \frac{d\varrho_1}{dr}, \tag{7}$$

where c is the sound velocity. Denoting by v_t the (one component) r. m. s turbulent velocity, we have

$$f_t = -\frac{d}{dr}\left(\frac{1}{3}\varrho_1 v_t^2\right). \tag{8}$$

The distances between the individual H II regions are great compared to their mean radius s. Therefore, the interaction between the neutral gas flow and the H II regions system can be represented as a sum of interactions between the neutral hydrogen and the individual H II regions. We admit this interaction to be proportional to the dynamic pressure and the cross-section of the region. The coefficient of proportionality β depends on the Mach number; its asymptotic value for a spherical region is 1.85 for large Mach numbers. Thus, we write

$$f_i = -\frac{3\beta}{8sN m_H}\varrho_1\varrho_2(v_r - u_r)\cdot|v_r - u_r|, \tag{9}$$

where N is the proton density inside the H II region.

The expression for the magnetic force density is

$$f_m = -\frac{H_\varphi}{4\pi r}\frac{d}{dr}(rH_\varphi).$$

Eliminating the magnetic field components with the aid of Eqs. (4) and (5) we obtain

$$f_m = -\frac{\Delta + \frac{u_\varphi}{r}\Gamma}{4\pi r u_r^3}\left[u_r\left(\Delta + \frac{du_\varphi}{dr}\Gamma\right) - r\frac{du_r}{dr}\left(\Delta + \frac{u_\varphi}{r}\Gamma\right)\right]. \tag{10}$$

In this manner we obtain a system of three nonlinear equations (1), (2), (3) with f's defined by (7), (8), (9), (10) for four functions: v_r, u_r, ϱ_1, ϱ_2. The ionization equation is the lacking one. To solve the problem we must know at least one of the functions and five boundary conditions: S, Δ, Γ, and two conditions for the equations of motion.

There is no possibility as yet to have any observational information on the boundary conditions for the Eqs. (2), (4), and (5). Fortunately, we know (roughly) from observations two functions: ϱ_1 and ϱ_2 (see Fig. 1). We can first, therefore, integrate the Eqs. (1) and (3) only, and then determine the constants Δ and Γ under the condition that the solution will satisfy as well as possible Eq. (2).

For numerical integrations we have adopted $\beta = 2$, $c = 1$ km/sec, $s = 4$ ps and $N = 10$ (Westerhout [6]), and ϱ_1, ϱ_2 from Fig. 1. The tur-

bulent velocity was taken according to table 6 of Schmidt [3] and K_ω from Schmidt [2]. Assuming $S = 7.4 \times 10^4$ cgs units and $v_r = 200$ km/sec for $r = 2$ kpc we obtain the best fit if

$$\Delta = -2 \times 10^{-3} \text{ gauss. km/sec},$$

$$\Gamma = 5.6 \times 10^{-5} \text{ gauss. kpc}.$$

In practice, the fitting was accomplished by varying Γ until the variance of Δ was minimum. The obtained value of Γ confirms the assumption of a strong radial magnetic field component.

Having the boundary values from the first integration ($u_r = 54$ km/sec for $r = 2.5$ kpc) we can now perform the second integration of the full system (1), (2), (3), considering as unknown the functions v_r, u_r, and ϱ_1. The observed values of ϱ_1 may be used as a test of our working model.

The results are represented on Figs. 1 (dotted line), 2 and 3. As we see, the proposed braking mechanism is sufficiently efficient to produce the sudden velocity decrease in the considered region. The magnetic field (the lines of force of which are unwinding spirals) is stronger than

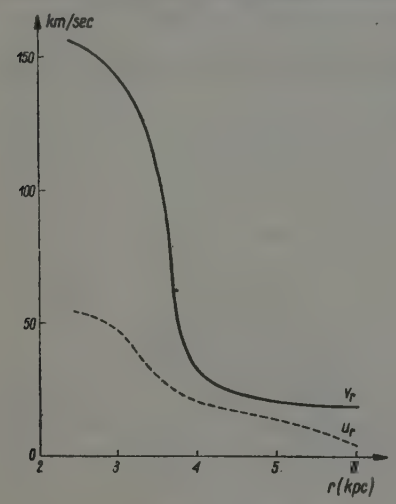

Fig. 1

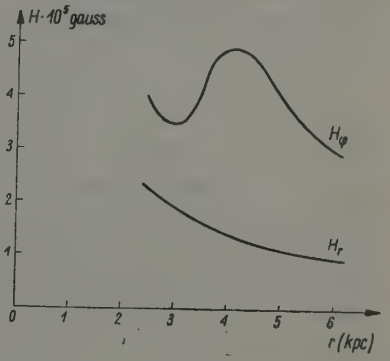

Fig. 2

Fig. 2. Expansion velocity decrease obtained from the second integration. Solid line — the velocity of neutral gas. Dashed line — the velocity of ionized gas.

Fig. 3. Radial H_r and transversal H_φ components of the magnetic field for $\Delta = -2 \times 10^{-3}$ gauss. km/sec and $\Gamma = 5.6 \times 10^{-5}$ gauss. kpc.

in the solar neighbourhood. Its intensity is determined by the momentum flux carried by the expanding gas.

The position of the computed ϱ_1-increase coincides very well with the observed one — in favour of our model. The height of the increase however, is over-estimated by a factor or two. If it were a result of omission of the star formation term in the continuity equation, the corresponding

net rate of star formation would be about 40 times the value of 9×10^{-12} $M_\odot$/cu. ps. yr. given by Schwarzschild [4] for the solar neighbourhood.

The numerical discrepancies are however of no great significance, most of the numerical starting data being extremely uncertain. Even the role of the basic physical processes is not quite clear. For instance, if the particle density of the H I ionized substratum were greater than 10^{-5}, the interaction of such a substratum with the neutral gas would be the dominant factor.

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Fraunhofer Diffraction of an Electromagnetic Wave

by

B. KARCZEWSKI

Presented by W. RUBINOWICZ on June 23, 1959

F. Kottler was the first to give [1] the correct formulation of Huygens' principle for electromagnetic waves. His treatment was not that of a boundary problem but of a saltus problem, assuming a known discontinuity of the components of the electromagnetic field at the surface of the diffracting screen. The formulae derived by Kottler account for diffraction of a dipole wave on the screen and for diffraction of a plane wave. In both cases the point of observation is at a finite distance from the diffracting screen, thus determining a Frensel type diffraction problem. This raises the question of adapting Kottler's theory to Fraunhofer diffraction, when both the source and observation points are at infinity. Of course, the starting point in the present investigation will be given by Kottler's formulas describing diffraction of an incident plane wave.

1. For a scalar spherical wave, Rubinowicz [2] basing on his well-known formula for the diffracted wave as determined by the curvilinear integral taken along the boundary of the diffracting aperture Γ, carried out the transition to Fraunhofer type diffraction by a limiting process. This was necessary as the diffracted wave originally derived by Rubinowicz [3], [4] accounted for Fresnel diffraction. It was also proved [2] that, in the case of Fraunhofer diffraction, the diffracted wave is continuous, when the vector in the direction of observation is antiparallel to the one determining the direction of propagation of the incident wave. This point will be discussed further in the present paper.

In Fresnel type diffraction phenomena, the diffracted wave was discontinuous at the boundary of the geometrical shadow (the corollary in Fraunhofer diffraction is provided by the foregoing antiparallel array of the vectors determining the direction of observation and that of propagation of the incident wave). However, the discontinuity of the diffracted wave is compensated at the shadow boundary by a discontinuity of the incident wave, resulting in continuous wave motion.

In the first place, the scalar case will be considered, wherein the plane wave $\exp(-ik\vec{R}\vec{p})$ is incident upon the diffracting aperture $\sum$ in the Kirchhoff screen; $\vec{R}$ denotes the radius vector and $\vec{p}$ is the unit vector in the direction of propagation of the wave. The diffraction field is given by the well-known formula due to Kirchhoff

$$u(P) = \frac{1}{4\pi} \int_{\Sigma} \left(e^{-ik\vec{R}\vec{p}} \frac{\partial}{\partial n} \frac{e^{-ikr}}{r} - \frac{e^{-ikr}}{r} \frac{\partial}{\partial n} e^{-ik\vec{R}\vec{p}} \right) d\Sigma , \tag{1}$$

where r is the distance of the point of observation P from a point on the surface $\sum$, and $\vec{n}$ is the unit vector normal to $\sum$. On carrying out the differentiations of Eq. (1) and neglecting the term in $1/r$ with respect to that in $-ik$, we have

$$u(P) = -\frac{ik}{4\pi} \int_{\Sigma} e^{-ik(\vec{R}\vec{p}+r)} \left(\vec{n}\frac{\vec{r}}{r} - \vec{n}\vec{p} \right) d\Sigma . \tag{1a}$$

Transition to the case of Fraunhofer diffraction is carried out as follows. In the exponential expression in Eq. (1a) $r = r_0 + \vec{R}\vec{a}$ is introduced, with $\vec{a}$ denoting the unit vector in the direction of observation, and r_0 – the distance of the point of observation P from the origin of the system of co-ordinates; for large values of r, $1/r_0 \approx 1/r$. In the expression in brackets, $\vec{r} = r\vec{a}$ is introduced. To fix the amplitude and phase of the field of diffraction, the right hand side of (1a) is divided by $\frac{\exp(-ikr_0)}{r_0}$, yielding

$$u^F = -\frac{ik}{4\pi}(\vec{a} - \vec{p}) \int_{\Sigma} e^{-ik\vec{R}(\vec{a}+\vec{p})} \vec{n}\, d\Sigma . \tag{2}$$

The foregoing formula describes the field of diffraction in Fraunhofer's case, u^F being a continuous function.

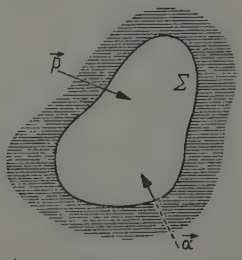

Fig. 1

If a plane wave is incident upon the diffracting aperture $\sum$, the diffracted wave u_d is given as follows (Rubinowicz [3], Fresnel diffraction):

$$u_d = -\frac{1}{4\pi} \oint_{\Gamma} \frac{e^{-ik(\vec{R}\vec{p}+r)}}{r} \frac{(\vec{p}\times\vec{r})\vec{ds}}{r + \vec{r}\vec{p}} . \tag{3}$$

Analogously, the following expression is derived from Eq. (3) for the wave diffracted in Fraunhofer type diffraction:

$$u_d^F = -\frac{1}{4\pi}\frac{\vec{p}\times\vec{a}}{1+\vec{p}\vec{a}}\oint_\Gamma e^{ik\vec{R}(\vec{a}+\vec{p})}\,\vec{ds}\,. \tag{4}$$

It will now be proved that $u_d^F = u^F$. To this effect, the formula $\oint_\Gamma \varphi\,\vec{ds} = -\int_\Sigma \operatorname{grad}\varphi\times\vec{n}\,d\Sigma$ will be applied, φ being assumed equal to $\exp[ik\vec{R}(\vec{a}+\vec{p})]$. The following formula is derived easily:

$$\oint_\Gamma e^{-ik\vec{R}(\vec{a}+\vec{p})}\,\vec{ds} = ik\int_\Sigma (\vec{p}+\vec{a})\times\vec{n}e^{-ik\vec{R}(\vec{a}+\vec{p})}d\Sigma\,. \tag{5}$$

Multiplying both sides of Eq. (5) by $\left(-\frac{1}{4\pi}\right)\cdot\frac{\vec{p}\times\vec{a}}{1+\vec{p}\vec{a}}$, using the relation $(\vec{a}\times\vec{b})(\vec{c}\times\vec{d}) = (\vec{a}\vec{c})(\vec{b}\vec{d})-(\vec{a}\vec{d})(\vec{b}\vec{c})$, and accounting for Eqs. (2), (4) proves $u_d^F = u^F$. In the limiting case $\vec{a} = -\vec{p}$, the continuous function u^F takes the value $u^F_{\vec{a}=-\vec{p}} = \frac{ik}{2\pi}\vec{p}\int_\Sigma \vec{n}\,d\Sigma$. In the case under consideration, by Eqs. (2), (4), (5), the limit value of the function u_d^F is exactly the same. If the value of the function u_d^F is now assumed in the case considered, to equal, *per definitionem*, its limit value, then the function u_d^F can be said to be a continuous function for any direction of observation. This is a conclusion that is correct for Fraunhofer diffraction only, since in the case of Fresnel diffraction

$$\frac{1}{4\pi}\int_\Sigma\left(e^{-ik\vec{R}\vec{p}}\frac{\partial}{\partial n}\frac{e^{-ikr}}{r}-\frac{e^{-ikr}}{r}\frac{\partial}{\partial n}e^{-ik\vec{R}\vec{p}}\right)d\Sigma$$
$$= \varepsilon e^{-ik\vec{R}\vec{p}} - \frac{1}{4\pi}\oint_\Gamma \frac{e^{-ik(\vec{R}\vec{p}+r)}}{r}\frac{(\vec{p}\times\vec{r})\,\vec{ds}}{r+\vec{p}\vec{r}}, \tag{6}$$

where ε takes the values 1 or 0 according to whether the point considered lies in the region of light or shadow.

2. Assume now the plane wave

$$\vec{E}_0(Q) = \vec{A}e^{-ik\vec{R}\vec{p}},$$
$$\vec{H}_0(Q) = (\vec{p}\times\vec{A})e^{-ik\vec{R}\vec{p}}$$

to be incident upon the diffracting aperture $\sum$ (Q denotes the variable point of the surface $\sum$, $\vec{A}$ — the amplitude vector). Kottler's formulas can be written as follows:

$$\vec{E}(P) = \frac{\vec{A}}{4\pi}\int_\Sigma\left(e^{-ik\vec{R}\vec{p}}\frac{\partial}{\partial n}\frac{e^{-ikr}}{r}-\frac{e^{-ikr}}{r}\frac{\partial}{\partial n}e^{-ik\vec{R}\vec{p}}\right)d\Sigma +$$
$$+\frac{1}{4\pi}\oint_\Gamma \vec{ds}\times\vec{A}\frac{e^{-ik(\vec{R}\vec{p}+r)}}{r}+$$
$$+\frac{1}{4\pi ik}\operatorname{grad}_P\oint_\Gamma \vec{ds}(\vec{p}\times\vec{A})\frac{e^{-ik(\vec{R}\vec{p}+r)}}{r}, \tag{7a}$$

$$\text{(7b)}\qquad \vec{H}(P)=\frac{\vec{p}\times\vec{A}}{4\pi}\int_{\Sigma}\left(e^{-ik\vec{R}\vec{p}}\frac{\partial}{\partial n}\frac{e^{-ikr}}{r}-\frac{e^{-ikr}}{r}\frac{\partial}{\partial n}e^{-ik\vec{R}\vec{p}}\right)d\Sigma+$$

$$+\frac{1}{4\pi}\oint_{\Gamma}\vec{ds}\times(\vec{p}\times\vec{A})\frac{e^{-ik(\vec{R}\vec{p}+r)}}{r}-$$

$$-\frac{1}{4\pi ik}\operatorname{grad}_P\oint_{\Gamma}(\vec{ds}\vec{A})\frac{e^{-ik(\vec{R}\vec{p}+r)}}{r}.$$

Consider the vector $\vec{E}(P)$. By analogy with the scalar case, the first right hand term will be denominated the Kirchhoff term, the second — the Larmor-Tedone term, and the third — the Kottler term. The terminology is related to the consecutive steps in the history of the formulation of Huygens' principle for electromagnetic waves (Baker and Copson [5]).

The transition with Eq. (7a) to the case of Fraunhofer type diffraction is carried out analogously as in deriving Eqs. (2), (4), the term in $1/r$ in the Kottler term (resulting from application of the operator grad_P) being neglected with respect to that in $-ik$. This yields

$$\text{(8)}\qquad \vec{E}^F=\frac{-ik}{4\pi}\vec{A}\left[(\vec{a}-\vec{p})\cdot\int_{\Sigma}e^{-ik\vec{R}(\vec{a}+\vec{p})}\vec{n}\,d\Sigma-\right.$$

$$-\frac{1}{4\pi}\vec{A}\times\oint_{\Gamma}e^{-ik\vec{R}(\vec{a}+\vec{p})}\vec{ds}+$$

$$\left.+\frac{1}{4\pi}\vec{a}\left[(\vec{p}\times\vec{A})\cdot\oint_{\Gamma}e^{-ik\vec{R}(\vec{a}+\vec{p})}\,ds\right.\right.\!.$$

Applying Eq. (5) to the two foregoing integrals, and with the notation $\vec{F}=\int_{\Sigma}\vec{n}\exp[-ik\vec{R}(\vec{a}+\vec{p})]d\Sigma$, we have

$$\text{(9a)}\qquad \vec{E}^F=-\frac{ik}{4\pi}\vec{A}[(\vec{a}-\vec{p})\vec{F}]-\frac{ik}{4\pi}\vec{A}\times[(\vec{a}+\vec{p})\times\vec{F}+$$

$$+\frac{ik}{4\pi}\vec{a}\,\{(\vec{p}\times\vec{A})\cdot[(\vec{a}+\vec{p})\times\vec{F}]\}.$$

Analogically, the following formula for $\vec{H}^F$ is obtained:

$$\text{(9b)}\qquad \vec{H}^F=\frac{-ik}{4\pi}(\vec{p}\times\vec{A})[(\vec{a}-\vec{p})\vec{F}]-\frac{ik}{4\pi}\{(\vec{p}\times\vec{A})\times[(\vec{a}+\vec{p})\times\vec{F}]\}-$$

$$-\frac{ik}{4\pi}\vec{a}\,\{\vec{A}\cdot[(\vec{a}+\vec{p})\times\vec{F}]\}.$$

Eqs. (9a, b) determine the vector field of diffraction in diffractional phenomena of the Fraunhofer type.

3. In analogy to the scalar case, the notion of diffracted wave can be introduced in Kottler's theory. This wave will be determined by Eqs. (7a, b) wherein, by Eq. (6), the integral over the surface $\sum$ is to be replaced by the curvilinear integral, and neglecting the factor $\exp(-ik\vec{R}\vec{p})$ accounting for the plane wave propagating, according to geometrical optics. With respect to Eqs. (7a, b), the foregoing definition yields

$$\vec{E}_d(P) = -\frac{\vec{A}}{4\pi}\oint_\Gamma \frac{e^{-ik(\vec{R}\vec{p}+r)}}{r}\frac{(\vec{p}\times\vec{r})\,\vec{ds}}{r+\vec{r}\vec{p}} +$$

$$+\frac{1}{4\pi}\oint_\Gamma \vec{ds}\times\vec{A}\,\frac{e^{-ik(\vec{R}\vec{p}+r)}}{r} + \frac{1}{4\pi ik}\operatorname{grad}_P \oint_\Gamma \vec{ds}(\vec{p}\times\vec{A})\frac{e^{-ik(\vec{R}\vec{p}+r)}}{r}, \tag{10}$$

$$\vec{H}_d(P) = -\frac{\vec{p}\times\vec{A}}{4\pi}\oint_\Gamma \frac{e^{-ik(\vec{R}\vec{p}+r)}}{r}\frac{(\vec{p}\times\vec{r})\,\vec{ds}}{r+\vec{r}\vec{p}} +$$

$$+\frac{1}{4\pi}\oint \vec{ds}\times(\vec{p}\times\vec{A})\frac{e^{-ik(\vec{R}\vec{p}+r)}}{r} - \frac{1}{4\pi ik}\operatorname{grad}_P \oint_\Gamma (\vec{ds}\,\vec{A})\frac{e^{-ik(\vec{R}\vec{p}+r)}}{r}.$$

From Eqs. (2), (3), (4), (5) it is to be concluded that, also in Kottler's vectorial theory generalized to include Fraunhofer diffraction, the diffracted wave given by the vectors $\vec{E}_d, \vec{H}_d$ is continuous provided the vectors $\vec{E}$, $\vec{H}$ at $\vec{a} = -\vec{p}$ are assumed to have their boundary values for $\vec{a} \to -\vec{p}$, as in the scalar case. This, in turn, leads to the conclusion that the electromagnetic wave propagating according to the laws of geometrical optics plays no part in Fraunhofer diffraction phenomena. Considering once more Eqs. (9a, b), it should be noted that if the direction of observation $\vec{a}$ is antiparallel to the vector $\vec{p}$, i. e. if $\vec{a} = -\vec{p}$, these equations take a much more simple form, reducing to

$$\vec{E}^P = \frac{ik}{2\pi}\vec{A}(\vec{p}\vec{F}'),$$

$$\vec{H}^F = \frac{ik}{2\pi}(\vec{p}\times\vec{A})(\vec{p}\vec{F}'); \qquad \vec{F}' = \int_\Sigma \vec{n}\,d\Sigma. \tag{11}$$

Thus, in the case under consideration, the vector field is determined by the Kirchhoff terms only, and Kirchhoff's scalar theory is in agreement with the vectorial one due to Kottler. The foregoing conclusion is also correct, if the direction $\vec{a}$ differs but little from $-\vec{p}$, as in this case the length of the vector $\vec{a}+\vec{p}$ is a very small quantity and it is the Kirchhoff term that contributes chiefly towards the values of the vectors $\vec{E}$ and $\vec{H}$.

The author expresses his indebtedness to Professor W. Rubinowicz, who formulated the subject of the present investigation, for his discussions and critical remarks.

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On the Generalization of the Variational Method of Bogoljubov

by

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In this note we generalize Bogoljubov's variational method of computation of the partition function. This method was applied without proof by Kvasnikov [1]-[3] to the problem of crystalline solids. We employed this method for computation concerning the problem of the lattice gas [4].

Following the idea of Hölder [5] we consider the $2n$-fold differentiable function $\varphi(t)$. We assume, that in this case $d^{2n}\varphi/dt^{2n} \geqslant 0$.

Thus, we can write

$$(1) \quad \varphi(t_\nu) = \varphi(M) + (t_\nu - M)\varphi'(M) + \ldots, + \frac{1}{(2n-1)!}(t_\nu - M)^{2n-1}\varphi^{(2n-1)}(M) + \frac{1}{(2n)!}(t_\nu - M)^{2n}\varphi^{(2m)}(\tau_\nu),$$

where

$$M = \Big(\sum_{\nu=1}^{N} t_\nu p_\nu\Big)\Big(\sum_{\nu=1}^{N} p_\nu\Big)^{-1}; \qquad p_\nu > 0$$

and

$$M \leqslant \tau_\nu \leqslant t_\nu \quad \text{if} \quad M < t_\nu,$$

$$t_\nu \leqslant \tau_\nu \leqslant M \quad \text{if} \quad t_\nu < M.$$

Multiplying (1) by p_ν and summing over ν we obtain

$$(2) \quad \sum_\nu p_\nu \varphi(t_\nu) \geqslant \varphi(M) \sum_\nu p_\nu + \frac{1}{2}\varphi''(M) \sum_\nu p_\nu (t_\nu - M)^2 + \ldots, + \frac{1}{(2n-1)!}\varphi^{(2n-1)}(M) \sum_\nu p_\nu (t_\nu - M)^{2n-1}.$$

Choosing: $\varphi(t) = e^{-\beta t}$, $p_\nu = e^{-\beta E_{0\nu}}$, $t_\nu = E'_\nu$, and $E_\nu = E_{0\nu} + E'_\nu$, we obtain *)

$$(3) \quad Z = \sum_\nu e^{-\beta E_\nu} \geqslant e^{-\beta M}\left[Z_0 + \frac{\beta^2}{2}\sum_\nu e^{-\beta E_{0\nu}}(M - E'_\nu)^2 + \right.$$

$$+ \frac{\beta^3}{3!}\sum_\nu e^{-\beta E_{0\nu}}(M - E'_\nu)^3 + , \ldots , +$$

$$\left. + \frac{\beta^{(2n-1)}}{(2n-1)!}\sum_\nu e^{-\beta E_{0\nu}}(M - E'_\nu)^{2n-1}\right] = Z_{\inf}^{[n]},$$

where $Z_0 = \sum_\nu e^{-\beta E_{0\nu}}$; $M = Z_0^{-1}\sum_\nu E'_\nu e^{-\beta E_{0\nu}}$.

If we choose $n = 1$, the inequality (3) yields

$$Z \geqslant Z_0 e^{-\beta M},$$

which is identical with the estimation given by Bogoljubov. Following Bogoljubov's method, we divide the energy E_ν into two parts: $E_{0\nu}$ and E'_ν, using the variational parameters. We maximalize the right side of (3) by means of differentiation with respect to variational parameters.

On the other hand, if the value $E_{\min}$ exists, we have

$$(4) \quad Z \leqslant Z_{\inf}^{[n]} + \frac{\beta^{2n}}{(2n)!} e^{-\beta E_{\min}} \sum_\nu (E'_\nu - M)^{2n} e^{-\beta E_{0\nu}},$$

because $E_{\min} \leqslant M \leqslant E_{\max}$, as can be easily checked (see e. g. [6]).

For the quantum-mechanical systems we can use the theorem given by Peierls [7]

$$(5) \quad Sp\, e^{-\beta \hat{H}} \geqslant \sum_\nu e^{-\beta \hat{H}_{\nu\nu}},$$

where $\hat{H}_{\nu\nu} = \langle \nu | \hat{H} | \nu \rangle$ and $|\nu\rangle$ is an arbitrary representation in the Hilbert space. For the sum on the right side of (5) we may employ our generalized variational method if necessary for reasons of computations.

To improve the estimation of the partition function given by Peierls by making use of nondiagonal elements of H a generalization of his theorem is needed.

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On the Possibility of Determining the Spin of ${}_{\Lambda}H^{4}$ from the Angular Distribution for Mesonic Decay

by

D. CHLEBOWSKA and J. SZYMAŃSKI

Presented by W. RUBINOWICZ on August 8, 1959

The problem of the spin of the hyperfragments ${}_{\Lambda}H^{4}$ and ${}_{\Lambda}He^{4}$ was first raised by Dalitz [1], who suggested a solution by investigating upward-downward asymmetry with respect to the normal to the plain of production of the hyperfragments in the angular distribution of the mesons. A difficulty arises, however, from the fact that the greater part of the hyperfragments analyzed come from plates wherein the plain of production is not generally known. Dalitz also suggested the possibility of determining the spin of a hyperfragment by way of investigating angular correlations of the products of decay with respect to the trajectory of motion of the hyperfragment. A similar effect was given an exact theoretical treatment by Chou and Shirokov [2], who carried out computations for the reaction

$$K^{-}+\mathrm{He}\rightarrow{}_{\Lambda}\mathrm{H}^{4}+\pi^{0}\rightarrow\mathrm{He}+\pi^{-}.$$

The discussion took account of the angle subtended by the two directions of the momentum of the mesons. For a totally polarized ${}_{\Lambda}H^{4}$ hyperfragment, the angular distribution was obtained in the form $W(\vartheta) = A\sin^{2}\vartheta$.

Data available at present are insufficient for determining the spin of hyperfragments from the energy [3]. It is always possible to choose the potential with a prevalent triplet or singlet state so as to obtain a convenient value of the binding energy within experimental error.

According to whether the spin of ${}_{\Lambda}H^{4}$ is 1 or 0, different values should result for the ratio

$$\frac{{}_{\Lambda}\mathrm{H}^{4}\rightarrow\mathrm{He}^{4}+\pi^{-}}{\text{the remaining channels of decay}}$$

Here, the difficulty resides in the fact that the various channels of decay are differently biased in scanning; this can give rise to error when comparing with experimental results.

It is the aim of the present paper to obtain some indications concerning the spin of the ${}_{\Lambda}H^4$ hyperfragment from the analysis of the energy and angular distribution for the three-body decay ${}_{\Lambda}H^4 \to t + p + \pi$. In the absence of interaction between the products of decay, the energy and angular distributions are independent of the spin of the hyperfragment and depend solely on the wave function $f(\vec{r})$ of the Λ particle therein (here, $\vec{r} = \vec{r}_{\Lambda} - \vec{r}_c$, with $\vec{r}_c$ and $\vec{r}_{\Lambda}$ denoting the co-ordinates of centre of mass of the core and that of the Λ particle, respectively). Differences in the energy and angular distributions can arise solely from differences in the interaction between the products of decay. That is why the case of H^4 presents advantages. Data concerning the scattering of protons on tritium point to a $p-t$ interaction that depends strongly on the kind of spin state present. In the singlet state the phase shifts are much lesser than in the triplet state.

If the decay of a Λ related to H^3 occurs in the s state, the proton and triton generated in the decay of ${}_{\Lambda}H^4$ are in the same state of spin as was the hyperfragment. This should give rise to a strong dependence of the angular distribution on the spin of ${}_{\Lambda}H^4$.

The hyperfragment ${}_{\Lambda}He^4$ should possess a spin identical to that of ${}_{\Lambda}H^4$; this results from charge invariance. In the case of ${}_{\Lambda}He^4$, no clear dependence of the distributions on the spin is to be expected since the difference in p — He the interaction in the singlet and triplet states is small.

It is essential to decide what is the spin of ${}_{\Lambda}H^4$ and ${}_{\Lambda}He^4$, as this may help to establish the parity of strange particles. If ${}_{\Lambda}H^4$ and ${}_{\Lambda}He^4$ possess zero spin, then the reactions

$$K^- + \alpha \to \begin{cases} {}_{\Lambda}H^4 + \pi^0 \\ {}_{\Lambda}He^4 + \pi^- \end{cases}$$

are possible in the case of pseudoscalar meson K at equal parity of Λ-N and in that of scalar K at different parity only.

It should be of some interest to determine the $\frac{p}{s}$ ratio for the decay of the Λ particle in the hyperfragment. This ratio may be different for a bound and a free Λ. Fig. 1 gives examples of graphs contributing to the polarisation of the meson shell surrounding the Λ, and is hence adapted to different renormalisations of the constants s and p with respect to the free Λ.

Dalitz estimated the $\frac{p}{s}$ ratio for a bound Λ from the branching ratio of mesonic to non-mesonic decay processes, obtaining results that

differ from those for a free Λ. The differences can be ascribed to deviations in the experimental values of the branching ratio arising from inhomogeneous scanning for the various channels of decay.

The ratio $\frac{p}{s}$ can also be evaluated from the angular distribution of the decay products of the hyperfragment, if interactions between the

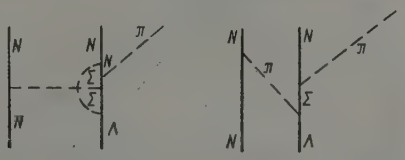

Fig. 1. Examples of graphs yielding deformation of mesonic shell of Λ particle

latter depend on the channel of decay of the Λ. The condition is not, however, fulfilled for all hyperfragments, e. g., in the case of $_{\Lambda}He^5$ the p-He^4 interaction is practically independent of the state wherein the Λ

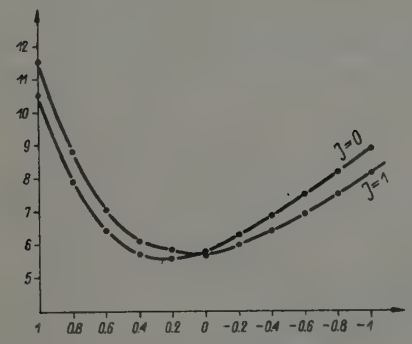

Fig. 2

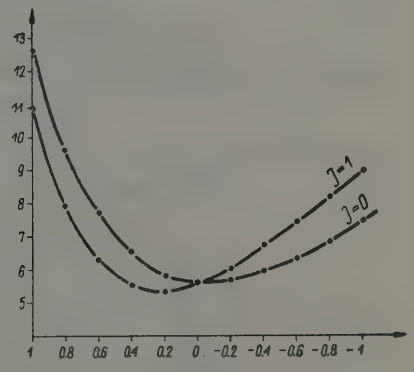

Fig. 3

Fig. 2. Angular distribution for decay in state s
Fig. 3. Angular distribution for decay in state p

decay occurred, thus yielding identical angular distributions. On the other hand, for $_{\Lambda}H^4$ the $p-t$ interaction is different for decay in the s and the p states, leading to different angular distributions and, hence, making it possible to evaluate $\frac{p}{s}$.

In the last part of the present paper, results of computations by the authors will be given showing the extent to which the ratio $\frac{p}{s}$ and the spin of $_{\Lambda}H^4$ can be assessed from the angular and energy distribution, accounting for interaction between the products of decay.

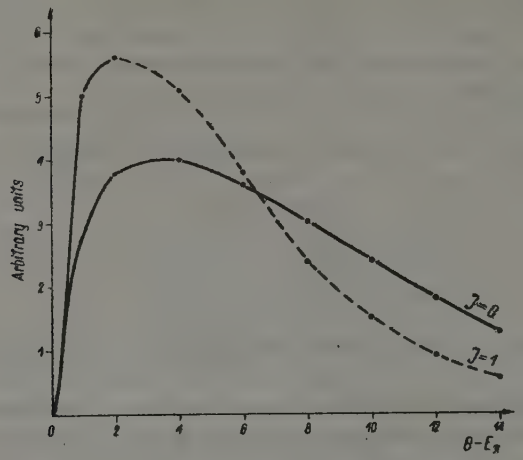

Fig. 4. Energy distribution for decayin state s

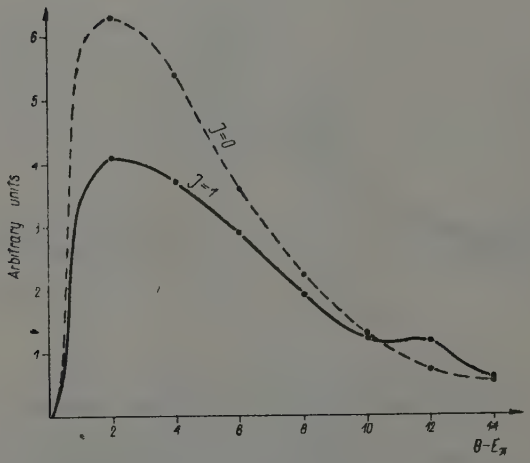

Fig. 5. Energy distribution for decay in state p

Computations

The computations were carried out by the J. Szymański method [6]; the respective paper also brings a discussion on the approximation applied here. The angular and energy distribution is computed as a function of the angle, between the direction of $\hat{K}$ (a versor having the direction of the π-meson) and $\hat{s}$ (the versor of the relative momentum of triton and proton). Computations are carried out separately for decay in the s and p states, since there is no interference between the two processes.

The probability for decay is given by formula (6) of J. Szymański's paper [6]

$$W = 2\pi\delta(E-E_0)|M|^2\,,$$

$$M = \langle\psi_f|\left(1+T\frac{1}{a}\right)H_1|\psi_0\rangle\,. \tag{1}$$

On introducing approximation discussed in the first part and in [6] we have

$$T = w + w\,\frac{1}{a+w}\,w\,,$$

wherein w denotes the sum of interactions of the π meason and of the proton with the core,

$$a = E - H_c - K_\pi - K_p + i\eta\,,$$

and K_π, K_p are the kinetic energies of the π meson and proton, respectively.

$$H_1 = g\int \psi_p(1+\beta\vec{\sigma}\vec{k})\psi_\wedge\cdot\varphi_\pi \tag{2}$$

$|\psi_0\rangle$ is assumed in the following form

$$|\psi_0\rangle = \sum_{\mu\eta}\int \frac{d_3p}{(2\pi)^3} f^{J\mathfrak{M}}_{\mu\eta}(\vec{p}_\wedge)|\vec{p}_\wedge\,\vec{K}\mu\eta\rangle \tag{3}$$

with μ denoting the spin of H^3, η — that of the Λ particle, and $\vec{p}_\wedge$ — the momentum of the latter with respect to the centre of mass of H^3.

ψ_f is the state of the free products of decay, $|\psi_f\rangle = |\vec{p}''\vec{q}''\vec{k}''\mu''\nu''\rangle$. Substituting $|\psi_0\rangle$ and $|\psi_f\rangle$ in (1) and introducing the unit operator T and H_1, we have

$$M = \langle\vec{p}''\vec{q}''\vec{k}''\mu''\nu''|H_1|\psi_0\rangle + \langle\vec{p}''\vec{q}''\vec{k}''\mu''\nu''|T|\vec{p}'\vec{q}'\vec{k}'\mu'\nu'\rangle\cdot\frac{1}{a(\vec{p}'\vec{q}'\vec{k}')}\,\langle\vec{p}'\vec{q}'\vec{k}'\mu'\nu'|H_1|\Psi_0\rangle \tag{4}$$

with $\vec{p}$, $\vec{q}$, $\vec{k}$ denoting the momentum of the triton, proton and pion, respectively, and ν being the projection of the spin of the proton. By (2),

$$\langle\vec{p}'\vec{q}'\vec{k}'\mu'\nu'|H_1|\Psi_0\rangle = f\left(\frac{1}{M}[m_\wedge\vec{p}' - m_t(\vec{q}'+\vec{k}')]\right)\sum_{\mu\eta}\delta_{\mu\mu'}\delta_{\eta\nu'}\times \tag{5}$$

$$\times\, c_{tt}(J\mathfrak{M}\mu\eta)\left[1+\beta k\sqrt{\frac{8\pi}{3}}(-1)^{t-\mu}c_{tt}(1\,M\mu-\nu)Y_{1M}\right]\delta(\vec{p}'+\vec{q}'+\vec{k}')\,,$$

$$\langle\vec{p}''\vec{q}''\vec{k}''\mu''\nu''|T|\vec{p}'\vec{q}'\vec{k}'\mu'\nu'\rangle = \langle\vec{s}\mu''\nu''|T|\vec{s}'\mu'\nu'\rangle(2\pi)^6\cdot\delta(\vec{k}''-\vec{k}')\,\delta(t''-t')\,. \tag{6}$$

Herein, the following notations have been introduced: $\vec{t} = \vec{p}+\vec{q}$, $\vec{s} = \frac{m_p}{M_1}\vec{p} - \frac{m_t}{M_1}\vec{q}$, $M_1 = m_p + m_t$, $M = m_p + m_t + m_\pi$.

From (6) we have further

$$\langle T\rangle = i\sum_{\substack{l\sigma\\ m\sigma_s}} Y_{lm}(\hat{s}')Y_{lm}(\hat{s}'')\sin\delta_{l\sigma}e^{i\delta l\sigma}\,c_{tt}(\sigma\sigma_s\mu'\nu')\cdot c_{tt}(\sigma\sigma_s\mu''\nu'')\,. \tag{7}$$

Introducing (5), (6) and (7) into Eq. (4) and neglecting the principal term yields the following expressions for M^s and M^p

$$M^s \sim \sum_{\mu\eta} f(\vec{x}-\vec{c})\,\delta_{\mu\mu''}\delta_{\eta\nu''}c_{tt}(J\mathfrak{M}\mu\eta) + i\sum_{lm\sigma_s}\int \delta\Omega_{\vec{s}'}f(\vec{x}-\vec{c})\times \tag{8}$$

$$\times\, Y^*_{lm}(\hat{s}')Y_{lm}(\hat{s}'')\sin\delta_{lJ}e^{i\delta lJ}c_{tt}(J\mathfrak{M}\mu''\nu'')\,.$$

and

$$M^p \sim k'' \sqrt{\frac{8\pi}{3}} \Big\{ \sum_{\eta} (-1)^{\frac{1}{2}-\nu''} c_{\frac{1}{2}\frac{1}{2}}(1M\nu''-\eta) c_{\frac{1}{2}\frac{1}{2}}(J\mathfrak{M}\mu''\eta) Y_{1M}(\hat{k}'') \times$$

$$\times f(\vec{x}-\vec{c}) + i \sum_{\substack{lm \\ \sigma_3\mu'\nu'}} (-1)^{\frac{1}{2}-\nu'} c_{\frac{1}{2}\frac{1}{2}}(\sigma\sigma_3\mu'\nu') c_{\frac{1}{2}\frac{1}{2}}(\sigma\sigma_3\mu''\nu'') c_{\frac{1}{2}\frac{1}{2}}(1M\nu'-\eta) \times \tag{9}$$

$$\times c_{\frac{1}{2}\frac{1}{2}}(J\mathfrak{M}\mu\eta) \delta_{\mu\mu'} Y_{lm}(\hat{s}') Y^*_{lm}(\hat{s}'') Y_{1M}(\hat{k}'') f(\vec{x}-\vec{c}) \sin\delta_{l\sigma} e^{i\delta_{l\sigma}},$$

with $\vec{x} = \frac{1}{M} m_\wedge \vec{p}''$, $\vec{c} = \frac{m}{M}(\vec{q}'' + \vec{k}'')$. From the fact that both in the angular and energy distribution the interference terms between the parts of different parity vanish, we have

$$W^s(x, c, \cos\delta) \sim (2J+1)\Big[F - \sum_{l_0} |M^s_{l_0}|^2 P_{l_0}(\cos\vartheta),$$

$$W^p(x, c, \cos\vartheta) \sim 2k''^2 \Big[\frac{2J+1}{2} F - \sum_{l_0} |M^p_{l_0}|^2 P_{l_0}(\cos\vartheta), \tag{10}$$

wherein $|M^s_{l_0}|^2$ and $|M^p_{l_0}|^2$ take the form:

$$|M^s_{l_0}|^2 = \sum_{ll'} f_l f_{l'} c^2_{ll'}(l_0 000)[2\sin^2\delta_{lJ} - \sin\delta_{lJ}\sin\delta_{l'J}\cos(\delta_{lJ}-\delta_{l'J})],$$

$$|M^p_{l_0}|^2 = \sum_{ll'\sigma} B(\sigma J) f_l f_{l'} c^2_{ll'}(l_0 000)[2\sin^2\delta_{lJ} - \sin\delta_{l\sigma}\sin\delta_{l'\sigma}\cos(\delta_{l\sigma}-\delta_{l'\sigma}), \tag{11}$$

$$F = f^2(\vec{x}-\vec{c}), \tag{12}$$

f_l being the coefficients of expansion of $f(\vec{x}-\vec{c})$ in Legendre polynomials normed as follows:

$$f_l = \frac{2l+1}{2} \int_{-1}^{1} f(\vec{x}\vec{c}) P_l(\cos(\vec{x}\vec{c})) d\cos(\vec{x}\vec{c}) \tag{13}$$

and

$$B(\sigma J) = (2\sigma+1)(2J+1) \begin{Bmatrix} 1 & J & \sigma \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix}.$$

The expressions for the angular energy distributions are obtained on integrating Eqs. (10) over auxiliary variables

$$W(\cos\vartheta) = \int W(x, c, \cos\vartheta)\delta(E-Q)x^2c^2\,dx\,dc,$$

$$W(k^2) = \int W(x, c, \cos\vartheta)\delta(E-Q)x^2\,dx\,d\cos\vartheta. \tag{14}$$

The results obtained on integrating Eqs. (14) and (15) numerically are shown in graphs 1 and 2; the curves obtained when no interaction is assumed are also shown for comparison.

Discussion

From Figs. 2, 3 and 4 it is seen that, while the distributions depend on the spin of ${}_\wedge H^4$ and the ratio $\frac{p}{s}$, the differences may be said to be small, so that ample statistics would be needed for final conclusions on

the spin and the ratio $\frac{p}{s}$. The authors refrain from including a comparison with the experimental results, as only several experimental cases were accessible. The number of cases may be expected to rise rapidly as a result of the use of bubble chambers.

From the graphs it is to be concluded that the ratio $\frac{p}{s}$ and the spin of ${}_{\Lambda}H^4$ cannot be determined simultaneously. If decay occurs in state s, the proton is scattered only in the spin state of the hyperfragment; if, on the other hand, it takes place in state p, scattering in the opposite state prevails. For this reason the curves $S = 1$ (decay in state s) and $S = 0$ (decay in state p) are almost identical.

Results obtained by the authors disclose the disadvantage of being dependent on the shape of the wave function of the hyperfragment. The dependence appears already at decay without final interactions.

The effect of the shape of the function may be largely eliminated if the computation is carried out in the system of the apparently resting Λ. In this case, the non-perturbated distribution does not depend on the shape of $f(\vec{r})$ and although it appears in the presence of interaction between the products of decay, it is probably much weaker than in the case considered. Computation in this system is, however, considerably involved.

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Оказывается, что независимость (в смысле работы [4]) в v-алгебрах обладает многими свойствами обычной линейной независимости: итак, имеют место теоремы о размерности, базисах и образующих, выполнена аксиома Витнея [9] о линейной независимости и т. д.

Разнообразные примеры v-алгебр и общая теорема о представлении этих алгебр даны К. Урбаником [8].

К. УРБАНИК, ТЕОРЕМА О ПРЕДСТАВЛЕНИИ АЛГЕБР МАРЧЕВСКОГО . . . . . стр. 617—619

Алгебры изучаемые Э. Марчевским (v-алгебры, см. работу Марчевского в настоящем выпуске) назовем *алгебрами Марчевского*. Здесь их примеры:

1. Пусть A — линейное пространство над полем K, A_0 — некоторое подпространство пространства A. Рассматриваем A как алгебру, в которой класс алгебраических функций состоит из всех функций вида $\lambda_1 x_1 + \ldots + \lambda_n x_n + a$, где коэффициенты λ_j принадлежат к K, переменные x_j пробегают A и свободный член a принадлежит к A_0.

2. Условия как в примере 1, причем коэффициенты λ_j удовлетворяют дополнительному условию $\lambda_1 + \ldots + \lambda_n = 1$.

3. Пусть A — произвольное множество и G — группа преобразований A в себя. Предполагаем, что любое не-тождественное преобразование $g \in G$ обладает не более, чем одной неподвижной точкой. Далее, предполагаем, что множество $B \subset A$ состоит из неподвижных точек и $g(B) \subset B$ для всех $g \in G$. Класс агебраических функций состоит из всех функций вида

$$f(x_1, \ldots, x_n) = g(x_j) \quad \text{или} \quad f(x_1, \ldots, x_n) = a\,,$$

где $g \in G$ и $a \in B$.

Теорема о представлении гласит, что *любая алгебра Марчевского является одной из алгебр перечисленных трех типов.*

Г. КРЕЙЗЕЛЬ, АНАЛИЗ ТЕОРЕМЫ КАНТОРА-БЕНДИКСОНА ПРОВЕДЕННЫЙ ПРИ ПОМОЩИ АНАЛИТИЧЕСКОЙ ИЕРАРХИИ стр. 621—626

В работе дается анализ теоремы Кантора-Бендиксона, проведенный с точки зрения аналитической иерархии определимых множеств. Замкнутое линейное множество определяется семейством G открытых интервалов таким, что $-F = \cup G$; если семейство G является рекуррентным (в очевидном смысле) либо принадлежит к одному из классов Π^1_k, Σ^1_k, тогда скажем, что F рекуррентно либо принадлежит к упомянутым классам Π^1_k, Σ^1_k.

Пусть $F = F_s \cup F_p$ — разложение F на часть рассеянную и часть совершенную. Автор приводит доказательство теоремы (доказанной также Лорентценом [4]), утверждающей, что если $F \in \Pi^1_1$, то $F_p \in \Pi^1_1$. Кромет ого, автором дается пример рекуррентного множества F такого, что F_p non-$\varepsilon\, \Sigma^1_1$. Эти результаты можно релятивизировать при применении аналитической иерархии по отношению к произвольному множеству K.

С. ГЖЕНДЗЕЛЬСКИЙ, ОБ ЭКСПАНСИИ ГАЗА ИЗ ВНУТРЕННИХ ЧАСТЕЙ ГАЛАКТИКИ . . . . . стр. 627—632

Обсуждается возможность объяснить внезапное задерживание экспансии газа из внутренних частей Галактики путем тормозящего действия на нейтральный водород облаков H II, вмороженных в осесимметрическое магнитное поле.

Оказывается, что трансверсальное поле порядка 10^{-5} гауссов может затормозить экспансию газа и привести к наблюдаемому увеличению плотности нейтрального водорода.

Б. КАРЧЕВСКИЙ, ФРАУНГОФЕРОВСКАЯ ДИФФРАКЦИЯ ЭЛЕКТРОМАГНИТНОЙ ВОЛНЫ . . . . . стр. 633—638

В работе дается распространение коттлеровской теории диффракции плоских электромагнитных волн на случай диффракции Фраунгофера. На основании полученных результатов автором доказано, что в граничном случае, т. е. когда вектор $\vec{\alpha}$, определяющий направление наблюдений, антипараллельный к вектору $\vec{p}$, определяющему направление падающей волны, — в формулах на напряжение поля остаются лишь выражения Кирхгоффа, тогда как выражения — поправки Лармора-Тедона и выражения Коттлера исчезают.

Если направление вектора $\vec{\alpha}$ лишь немногим отличается от направления вектора $-\vec{p}$, мы можем пренебречь этими поправками, так как они малы.

Итак, в обоих рассматриваемых случаях электромагнитная теория приводит к таким-же заключениям, что и скалярная теория диффракции.

Вводится понятие электромагнитной согнутой волны и доказывается, что для случая $\vec{\alpha} = -\vec{p}$ такая волна непрерывна.

Г. ЧЕРВОНКО, ОБОБЩЕНИЕ ВАРИАЦИОННОГО МЕТОДА БОГОЛЮБОВА . . . . . стр. 639—641

В работе излагается обобщение известного вариационного метода исчисления статистических сумм, предложенного Боголюбовым. При помощи этого метода можно ограничить статистическую сумму снизу. Такую оценку получаем путем разложения в ряд Тайлора экспоненциальной функции, выступающей в виде суммы.

В работе производится также оценка статистической суммы сверху в случае, в котором имеется наиболее низкое значение энергии.

Обобщенный вариационный метод можно применить также к квантовым системам, используя предварительно теоремы Пейерльса.

Д. ХЛЕБОВСКАЯ и И. ШИМАНСКИЙ, О ВОЗМОЖНОСТИ ОПРЕДЕЛЕНИЯ СПИНА ${}_{\Lambda}H^{4}$ УГЛОВОГО РАСПРЕДЕЛЕНИЯ (РАЗЛОЖЕНИЯ) ДЛЯ МЕЗОННОГО РАСПАДА . . . . . стр. 643—649

В работе рассматривается возможность определения спина гиперфрагментов ${}_{\Lambda}H^{4}$ и ${}_{\Lambda}He^{4}$ на основании угловых и энергетических распределений раз-

ложений для мезонных распадов. Учитывая взаимодействия между продуктами распада ($p - H^3$) выступают разницы между угловыми, а также энергетическими распределениями (разложениями) для спина ${}_{\Lambda}H^4$, равняющегося нулю и единице. При соответственно большой статистике эти разницы могут позволить произвести оценку спина ${}_{\Lambda}H^4$.

Одновременно рассматривается возможность определения отношения p/s; это отношение можно вычислить на основании угловых распределений (разложений) в таком случае, если спин ${}_{\Lambda}H^4$ известен на основании иных проведенных исследований.

В расчетах применялась импульсная приближенность.

Г. ЛОЖИКОВСКИЙ и М. РОЗВАДОВСКИЙ, ВЛИЯНИЕ УЛЬТРАЗВУКОВ НА ЯВЛЕНИЕ ЭЛЕКТРОЛЮМИНЕСЦЕНЦИИ СОЕДИНЕНИЯ ZnS : Cu . . . . . стр. 651–652

В работе даются результаты исследований, в которых получены изменения формы волн яркости электролюминесценции фосфора ZnS : Cu под влиянием поля ультразвуков для различных частот возбуждающего напряжения. Частота ультразвуковой волны равнялась 680 *кгц*.

Приводятся осциллограммы волн яркости электролюминесценции, иллюстрирующие их изменения под влиянием поля ультразвуков. Это явление не обнаруживает инертности, а следовательно можно предполагать, что оно не является эффектом изменений температуры.

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Influence of Ultrasonics on the Electroluminescence of ZnS : Cu

by

H. ŁOŻYKOWSKI and M. ROZWADOWSKI

Presented by A. JABŁOŃSKI on July 18, 1959

In recent years a number of papers dealing with the influence of ultrasonic fields on the photoluminescence [1]-[4] and conductivity of semiconductors [5], [6] have been published.

The present communication brings the results of observations made when investigating the effect of an ultrasonic field on the brightness waves of the electroluminescence of a ZnS:Cu phosphor (activator concentration being 4.6×10^{-3} g/g). For exciting electroluminescence, a sine field was used whose frequency varied from 200 Hz to 5 kHz. The brightness waves were observed by means of a photomultiplier on an oscillograph screen. The way ultrasonic energy was fed to the electroluminescence cell is shown in Fig. 1.

A barium titanate plate ($BaTiO_3$) was used as ultrasonic source. The frequency of the ultrasonic wave was 680 kHz. The electroluminophor consisted of a suspension of ZnS:Cu in castor oil. The upper face of "cone" *S* and the conducting glass plate (Fig. 1) were used as electrodes.

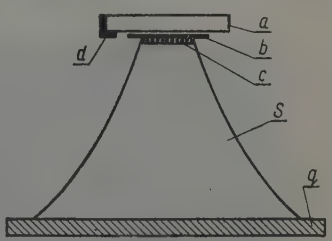

Fig. 1.

a — transparent conducting plate,
b — mica,
c — electroluminophor,
S — brass cone,
q — barium titanate plate.

Fig. 2 shows the influence of ultrasonic waves on the electroluminescence of the investigated phosphor. The oscillograms (Fig. 2*a*, 2*c*, 2*e*, 2*g*) show the brightness waves of electroluminescence in the absence of the ultrasonic field, whereas others (Fig. 2*b*, 2*d*, 2*f*, 2*h*) represent the brightness waves recorded when the ultrasonic field acts on the electroluminescence cell. The oscillograms were obtained at a frequency of 600 Hz (Fig. 2*a*, 2*b*), 1000 Hz (Fig. 2*c*, 2*d*), 1500 Hz (Fig. 2*e*, 2*f*) and

2000 Hz (Fig. 2*g*, 2*h*). It was observed that variations in the brightness appear immediately on switching on the ultrasonic field and disappeared as soon as the field was removed. This fact indicates that the effect is not due to variations in temperature, a point of view corroborated by findings (cf. [7]) that the temperature dependence of the brightness waves differs in shape from the authors results.

The authors are indebted to Professor A. Jabłoński for his interest and advice in their work.

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PLATE

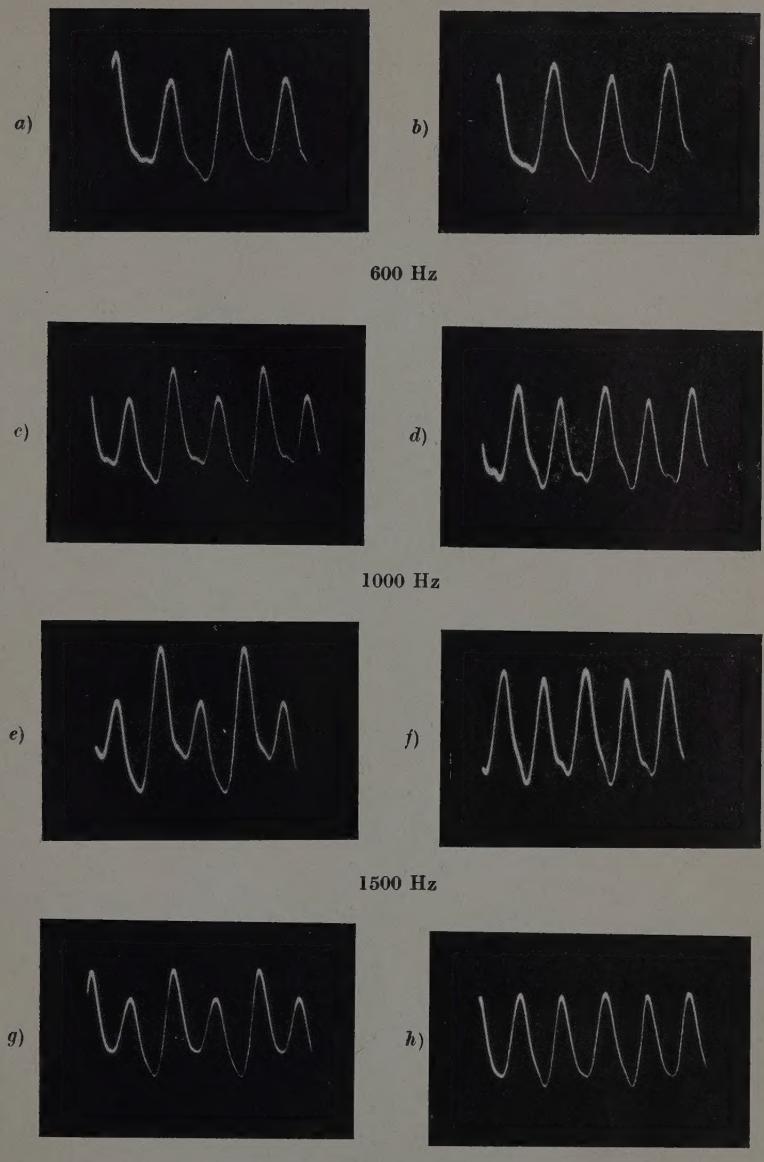

Fig. 2. Brightness waves of electroluminescence in ZnS:Cu.
b, d, f, h — phosphor within ultrasonic field,
a, c, e, g — ultrasonic field absent.

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VII 1959 ВЫПУСК 10

Ю. МЕДЕР, О НЕКОТОРОМ СООТНОШЕНИИ МЕЖДУ МЕТОДАМИ ЭЙЛЕРА-КНОППА И ЦЕЗАРО ОТНОСИТЕЛЬНО СУММИРУЕМОСТИ ОРТО-НОРМИРОВАННЫХ РЯДОВ . . . . . стр. 589–592

В статье доказываются две теоремы, из которых первая касается проблемы эквивалентности метода Эйлера-Кноппа и метода Цезаро относительно суммируемости орто-нормированных рядов. Другая теорема является обобщением результатов, полученных автором [1].

Пусть

$$\sum_{n=a}^{\infty} a_n \varphi_n^{(x)} \tag{1}$$

орто-нормированный ряд, при условии, что

$$\sum_{n=0}^{\infty} a_n^2 < \infty. \tag{2}$$

Справедлива следующая

Теорема 1. *В отнесении к суммируемости орто-нормированных рядов* (1), *при условии* (2), *метод* (C, r) *Цезаро существенно* (*т. е. кроме множества меры нуль*) *слабее, чем метод* (E, q) *Эйлера-Кноппа, для любого положительного порядка* q *и* r.

В доказательстве этой теоремы автор опирается на теорему Тандори [2] и на известную теорему Таубера [3].

Пусть $C^r - \lim s_n(x)$ и $E^q C^r - \lim s_n(x)$ обозначают обобщенные пределы последовательности частных сумм $s_n(x)$ ряда (1), полученных путем трансформации соответственно C^r и $E^q C^r$.

При этих обозначениях доказывается следующая

Теорема 2. *Если при данных, положительных* r *и* q *существует почти всюду* $E^q C^r - \lim s_n(x)$, *тогда существует почти всюду* $C^r - \lim s_n(x)$ *и* $E^q C^r - \lim s_n(x) = C^r - \lim s_n(x)$, *для всех* $r > 0$.

В доказательстве этой теоремы можно принять $r > 1$, что непосредственно вытекает из соотношения

$$E^q C^{r+1} \sim E^q C(C^r) = CE^q(C^r) \sim C(E^q C^r).$$

Итак, получаем наше утверждение из цитированной выше теоремы Таубера и леммы и теоремы Качмажа-Зыгмунда [7].

Г. МИЛИЦЕР-ГРУЖЕВСКАЯ, ТЕОРЕМА ОБ ОДНОЗНАЧНОСТИ РЕШЕНИЯ ПАРАБОЛИЧЕСКОЙ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ ЛИНЕЙНЫХ УРАВНЕНИЙ С КОЭФФИЦИЕНТАМИ ГЕЛЬДЕРА . . стр. 593—599

В работе доказывается теорема об однозначности решения параболической системы дифференциальных линейных уравнений с частными производными и с коэффициентами Гельдера в классе решений, которые совместно со всеми своими производными порядка, не превышающего порядка системы, интегрируемы абсолютно во времени-пространстве. Временем-пространством называем здесь декартово произведение эвклидова пространства и конечного времени.

М. МОШИНСКАЯ, ОБОБЩЕНИЕ ФОРМУЛЫ ГАУССА-БОННЕТА . . . . . стр. 601—609

В работе содержится обобщение на поверхности с некоторыми особенностями формулы Гаусса Боннета из дифференциальной геометрии.

Автор доказала, что если Δ является односвязным куском поверхности класса C^2 в каждой точке за исключением конечного числа точек и конечного числа регулярных дуг, не имеющих общих точек, то

$$\iint_{\Delta} K\,dS = \Big[2\pi - \sum_{j=1}^{n}(\pi - a_j) - \int_{\mathrm{Fr}\,\Delta} \varkappa_g\,ds\Big] - \sum_{i=1}^{m}(2\pi - \varphi_i) - \sum_{ij}\int_{L_i^j}(\varkappa_g^1 + \varkappa_g^2)\,ds\,,$$

где K — кривизна Гаусса на Δ, a_j $(j = 1, \ldots, n)$ — внутренние углы кривой $\Gamma = \mathrm{Fr}\,\Delta$, $\varphi_i (i = 1, \ldots, m)$ — полные углы вокруг точек у вершин куска Δ, $\varkappa_g^v$ $(v = 1, 2)$ — геодезическая кривизна ребра L_i^j на куске Q^i, который является продолжением куска Δ_j^i, за L_j, $(L_j = \bar{\Delta}_j^1 \frown \bar{\Delta}_j^2)$.

Э. МАРЧЕВСКИЙ, НЕЗАВИСИМОСТЬ В НЕКОТОРЫХ АБСТРАКТНЫХ АЛГЕБРАХ . . . . . стр. 611—616

Алгебра (см. определение в статье [4]) называется *v-алгеброй*, если она удовлетворяет условию: для любой меры алгебраических операций f и g от одинакового числа переменных, если равенство

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)\,,$$

(рассматриваемое как функция высказываний от n переменных) зависит от переменной x_1, то существует такая алгебраическая функция h, что последнее равенство эквивалентно следующему

$$x_1 = h(x_2, \ldots, x_{n-1})\,.$$